AA242B: MECHANICAL VIBRATIONS

Direct Time-Integration Methods

These slides are based on the recommended textbook: M. Géradin and D. Rixen, "Mechanical Vibrations: Theory and Applications to Structural Dynamics," Second Edition, Wiley, John & Sons, Incorporated, ISBN-13:9780471975465

Outline

1. Stability and Accuracy of Time-Integration Operators
2. Newmark's Family of Methods
3. Explicit Time Integration Using the Central Difference Algorithm

- Lagrange's equations of dynamic equilibrium ($\mathbf{p}(t) = \mathbf{0}$)

$$\begin{aligned} \mathbf{M}\ddot{\mathbf{q}} + \mathbf{C}\dot{\mathbf{q}} + \mathbf{K}\mathbf{q} &= \mathbf{0} \\ \mathbf{q}(0) &= \mathbf{q}_0 \\ \dot{\mathbf{q}}(0) &= \dot{\mathbf{q}}_0 \end{aligned} \tag{1}$$

- First-order form

$$\underbrace{\begin{pmatrix} \mathbf{0} & \mathbf{M} \\ \mathbf{M} & \mathbf{C} \end{pmatrix}}_{\mathbf{A}_B} \underbrace{\begin{pmatrix} \ddot{\mathbf{q}} \\ \dot{\mathbf{q}} \end{pmatrix}}_{\dot{\mathbf{u}}} + \underbrace{\begin{pmatrix} -\mathbf{M} & \mathbf{0} \\ \mathbf{0} & \mathbf{K} \end{pmatrix}}_{-\mathbf{A}_A} \underbrace{\begin{pmatrix} \dot{\mathbf{q}} \\ \mathbf{q} \end{pmatrix}}_{\mathbf{u}} = \underbrace{\begin{pmatrix} \mathbf{0} \\ \mathbf{0} \end{pmatrix}}_{\mathbf{0}}$$

$$\Longrightarrow \boxed{\dot{\mathbf{u}} = \mathbf{A}\mathbf{u}} \qquad \text{where} \quad \mathbf{A} = \mathbf{A}_B^{-1}\mathbf{A}_A$$

- Direct time-integration

- General multistep time-integration method for first-order systems of the form $\dot{\mathbf{u}} = \mathbf{A}\mathbf{u}$

$$\mathbf{u}_{n+1} = \sum_{j=1}^{m} \alpha_j \mathbf{u}_{n+1-j} - h \sum_{j=0}^{m} \beta_j \dot{\mathbf{u}}_{n+1-j}$$

where $h = t_{n+1} - t_n$ is the computational time-step, $\mathbf{u}_n = \mathbf{u}(t_n)$, and

$$\mathbf{u}_{n+1} = \begin{bmatrix} \dot{\mathbf{q}}_{n+1} \\ \mathbf{q}_{n+1} \end{bmatrix}$$

is the state-vector calculated at t_{n+1} from the m preceding state vectors and their derivatives as well as the derivative of the state-vector at t_{n+1}

  - $\beta_0 \neq 0$ leads to an **implicit** scheme — that is, a scheme where the evaluation of $\mathbf{u}_{n+1}$ requires the solution of a system of equations
  - $\beta_0 = 0$ corresponds to an **explicit** scheme — that is, a scheme where the evaluation of $\mathbf{u}_{n+1}$ does not require the solution of any system of equations and instead can be deduced directly from the results at the previous time-steps

- General multistep integration method for first-order systems (continue)

$$\mathbf{u}_{n+1} = \sum_{j=1}^{m} \alpha_j \mathbf{u}_{n+1-j} - h \sum_{j=0}^{m} \beta_j \dot{\mathbf{u}}_{n+1-j}$$

  - trapezoidal rule (implicit)

$$\mathbf{u}_{n+1} = \mathbf{u}_n + \frac{h}{2}(\dot{\mathbf{u}}_n + \dot{\mathbf{u}}_{n+1}) \Rightarrow (\frac{h}{2}\mathbf{A} - \mathbf{I})\mathbf{u}_{n+1} = -\mathbf{u}_n - \frac{h}{2}\dot{\mathbf{u}}_n$$

  - backward Euler formula (implicit)

$$\mathbf{u}_{n+1} = \mathbf{u}_n + h\dot{\mathbf{u}}_{n+1} \Rightarrow (h\mathbf{A} - \mathbf{I})\mathbf{u}_{n+1} = -\mathbf{u}_n$$

  - forward Euler formula (explicit)

$$\mathbf{u}_{n+1} = \mathbf{u}_n + h\dot{\mathbf{u}}_n \Rightarrow \mathbf{u}_{n+1} = (\mathbf{I} + h\mathbf{A})\mathbf{u}_n$$

- Consider an undamped one-degree-of-freedom oscillator

$$\ddot{q} + \omega_0^2 q = 0$$

with $\omega_0 = \pi$ rad/s and the initial displacement

$$q(0) = 1, \ \dot{q}(0) = 0$$

  - exact solution

$$q(t) = \cos \omega_0 t$$

  - associated first-order system

$$\dot{\mathbf{u}} = \mathbf{A}\mathbf{u}$$

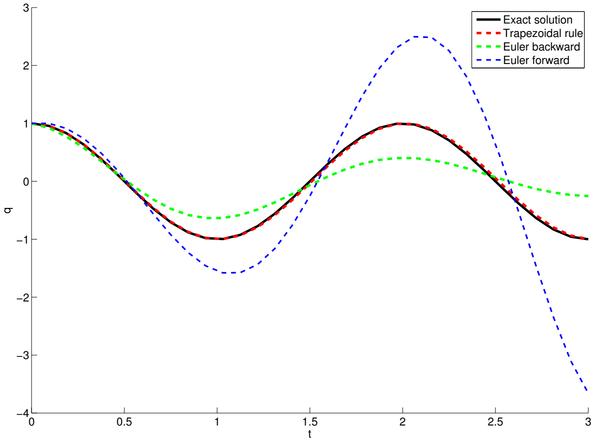
where

$$\mathbf{A} = \begin{bmatrix} 0 & -\omega_0^2 \\ 1 & 0 \end{bmatrix}$$

$\mathbf{u} = [\dot{q}, q]^T$, and initial condition

$$\mathbf{u}(0) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

- Numerical solution
  $T = 3s, \ h = \frac{T}{32}$



- Analysis of the characteristic equation of a time-integration method
  - consider the first-order system $\dot{\mathbf{u}} = \mathbf{A}\mathbf{u}$
  - for this problem, the general multistep method can be written as

$$\mathbf{u}_{n+1} = \sum_{j=1}^{m} \alpha_j \mathbf{u}_{n+1-j} - h \sum_{j=0}^{m} \beta_j \dot{\mathbf{u}}_{n+1-j} \Rightarrow \sum_{j=0}^{m} \left[\alpha_j \mathbf{I} - h\beta_j \mathbf{A}\right] \mathbf{u}_{n+1-j} = 0, \quad \alpha_0 = -1$$

  - let $\{\mu_r\}_{r=1}^{r=n}$ be the eigenvalues of $\mathbf{A}$ and $\mathbf{X}$ be the matrix of associated eigenvectors $\left(\mathbf{X}^{-1}\mathbf{A}\mathbf{X} = \text{diag}(\mu_1, \cdots, \mu_r, \cdots, \mu_n)\right)$
  - the characteristic equation associated with $\sum_{j=0}^{m} \left[\alpha_j \mathbf{I} - h\beta_j \mathbf{A}\right] \mathbf{u}_{n+1-j} = 0$ is obtained by searching for a solution of the form

$$\begin{aligned}
\mathbf{u}_{n+1-m} &= \mathbf{X}\mathbf{a} \quad (\textit{decomposition on an eigen basis}) \\
\mathbf{u}_{(n+1-m)+1} &= \lambda \mathbf{u}_{n+1-m} = \lambda \mathbf{X}\mathbf{a} \quad (\textit{solution form}) \\
&\vdots \\
\mathbf{u}_{n+1} &= \lambda \mathbf{u}_n = \cdots = \lambda^k \mathbf{u}_{n+1-k} = \cdots = \lambda^m \mathbf{X}\mathbf{a}
\end{aligned}$$

where $\lambda \in \mathbb{C}$ is called the solution amplification factor

- Analysis of the characteristic equation of a time-integration method (continue)
  - hence

$$\sum_{j=0}^{m} \left[\alpha_j \mathbf{I} - h\beta_j \mathbf{A}\right] \lambda^{m-j} \mathbf{X}\mathbf{a} = \mathbf{0}$$

  - since $\mathbf{X}^{-1}\mathbf{A}\mathbf{X} = \text{diag}(\mu_1, \cdots, \mu_r, \cdots, \mu_n)$, premultiplying the above result by $\mathbf{X}^{-1}$ leads to

$$\left[\sum_{j=0}^{m} \left[\alpha_j \mathbf{I} - h\beta_j \text{diag}(\mu_1, \cdots, \mu_r, \cdots, \mu_n)\right] \lambda^{m-j}\right] \mathbf{a} = \mathbf{0}$$

$$\Longrightarrow \boxed{\sum_{j=0}^{m} \left[\alpha_j - h\beta_j \mu_r\right] \lambda^{m-j} = 0, \;\; r = 1, 2, \cdots, n}$$

  - hence, the numerical response $\mathbf{u}_{n+1} = \lambda^m \mathbf{X}\mathbf{a}$ remains bounded if each solution of the above characteristic equation of degree m satisfies $|\lambda_k| < 1, \; k = 1, \cdots, m$

- Analysis of the characteristic equation of a time-integration method (continue)
  - the stability limit is a circle of unit radius
  - in the complex plane of $\mu_r h$, the stability limit is therefore given by writing $\lambda = e^{i\theta}, 0 \leq \theta \leq 2\pi$

$$\Longrightarrow \mu_r h = \frac{\sum_{j=0}^{m} \alpha_j e^{i(m-j)\theta}}{\sum_{j=0}^{m} \beta_j e^{i(m-j)\theta}}$$

  - one-step schemes $(m = 1)$

$$\mu_r h = \frac{\alpha_0 e^{i\theta} + \alpha_1}{\beta_0 e^{i\theta} + \beta_1} = \frac{-e^{i\theta} + \alpha_1}{\beta_0 e^{i\theta} + \beta_1}$$

- Analysis of the characteristic equation of a time-integration method (continue)
  - one-step schemes ($m = 1$) (continue)

$$\mu_r h = \frac{\alpha_0 e^{i\theta} + \alpha_1}{\beta_0 e^{i\theta} + \beta_1} = \frac{-e^{i\theta} + \alpha_1}{\beta_0 e^{i\theta} + \beta_1}$$

    - forward Euler: $\alpha_1 = 1$, $\beta_0 = 0$, $\beta_1 = -1 \Rightarrow \mu_r h = e^{i\theta} - 1$
      the solution is *unstable in the entire plane except inside the circle of unit radius and center* -1
    - backward Euler: $\alpha_1 = 1$, $\beta_0 = -1$, $\beta_1 = 0 \Rightarrow \mu_r h = 1 - e^{-i\theta}$
      the solution is *stable in the entire plane except inside the circle of unit radius and center* 1
    - trapezoidal rule: $\alpha_1 = 1$, $\beta_0 = -\frac{1}{2}$, $\beta_1 = -\frac{1}{2} \Rightarrow \mu_r h = \frac{2i\sin\theta}{1 + \cos\theta}$
      the solution is *stable in the entire left-hand plane*

- Analysis of the characteristic equation of a time-integration method (continue)
  - application to the single degree-of-freedom oscillator

$$\ddot{q} + \omega_0^2 q = 0, \qquad \mathbf{A} = \begin{bmatrix} 0 & -\omega_0^2 \\ 1 & 0 \end{bmatrix}$$

    - the eigenvalues are $\mu_r = \pm i\omega_0$
    - the roots $\mu_r h$ are located in the unstable region of the forward Euler scheme $\Rightarrow$ amplification of the numerical solution
    - the roots $\mu_r h$ are located in the stable region of the backward Euler scheme $\Rightarrow$ decay of the numerical solution
    - the roots $\mu_r h$ are located on the stable boundary of the trapezoidal rule scheme $\Rightarrow$ the amplitude of the oscillations is preserved

- Taylor's expansion of a function f

$$f(t_n+h) = f(t_n) + hf'(t_n) + \frac{h^2}{2}f''(t_n) + \cdots + \frac{h^s}{s!}f^{(s)}(t_n) + \frac{1}{s!}\int_{t_n}^{t_n+h} f^{(s+1)}(\tau)(t_n+h-\tau)^s d\tau$$

- Application to the velocities and displacements

$$\begin{array}{lclcl} f=\dot{\mathbf{q}},\ s=0 & \Rightarrow & \dot{\mathbf{q}}_{n+1} & = & \dot{\mathbf{q}}_n + \displaystyle\int_{t_n}^{t_{n+1}} \ddot{\mathbf{q}}(\tau)d\tau \\ f=\mathbf{q},\ s=1 & \Rightarrow & \mathbf{q}_{n+1} & = & \mathbf{q}_n + h\,\dot{\mathbf{q}}_n + \displaystyle\int_{t_n}^{t_{n+1}} \ddot{\mathbf{q}}(\tau)(t_{n+1}-\tau)d\tau \end{array} \qquad (2)$$

- Given
  - any approximation $\bar{\ddot{\mathbf{q}}}(\tau)$ of $\ddot{\mathbf{q}}(\tau)$ in the time-interval $[t_n,\ t_{n+1}]$ and any pair of quadrature rules for approximating the resulting integrals $\displaystyle\int_{t_n}^{t_{n+1}} \bar{\ddot{\mathbf{q}}}(\tau)d\tau$ and $\displaystyle\int_{t_n}^{t_{n+1}} \bar{\ddot{\mathbf{q}}}(\tau)(t_{n+1}-\tau)d\tau$
  - or any pair of direct approximations of the time-integrals $\displaystyle\int_{t_n}^{t_{n+1}} \ddot{\mathbf{q}}(\tau)d\tau$ and $\displaystyle\int_{t_n}^{t_{n+1}} \ddot{\mathbf{q}}(\tau)(t_{n+1}-\tau)d\tau$

(9) leads to a numerical time-integration scheme for solving (1)

- Taylor expansions of $\ddot{\mathbf{q}}_n$ and $\ddot{\mathbf{q}}_{n+1}$ around $\tau \in [t_n, t_{n+1}]$

$$\begin{aligned}
\ddot{\mathbf{q}}_n &= \ddot{\mathbf{q}}(\tau) + \mathbf{q}^{(3)}(\tau)(t_n - \tau) + \mathbf{q}^{(4)}(\tau)\frac{(t_n - \tau)^2}{2} + \cdots && (3) \\
\ddot{\mathbf{q}}_{n+1} &= \ddot{\mathbf{q}}(\tau) + \mathbf{q}^{(3)}(\tau)(t_{n+1} - \tau) + \mathbf{q}^{(4)}(\tau)\frac{(t_{n+1} - \tau)^2}{2} + \cdots && (4)
\end{aligned}$$

- Combine $(1-\gamma)$ (3) + γ (4) and extract $\ddot{\mathbf{q}}(\tau)$

$$\Longrightarrow \ddot{\mathbf{q}}(\tau) = (1-\gamma)\ddot{\mathbf{q}}_n + \gamma\ddot{\mathbf{q}}_{n+1} + \mathbf{q}^{(3)}(\tau)(\tau - h\gamma - t_n) + \mathcal{O}(h^2\mathbf{q}^{(4)})$$

- Combine $(1-2\beta)$ (3) + 2β (4) and extract $\ddot{\mathbf{q}}(\tau)$

$$\Longrightarrow \ddot{\mathbf{q}}(\tau) = (1-2\beta)\ddot{\mathbf{q}}_n + 2\beta\ddot{\mathbf{q}}_{n+1} + \mathbf{q}^{(3)}(\tau)(\tau - 2h\beta - t_n) + \mathcal{O}(h^2\mathbf{q}^{(4)})$$

- Substitute the 1^{st} expression of $\ddot{\mathbf{q}}(\tau)$ in $\int_{t_n}^{t_{n+1}} \ddot{\mathbf{q}}(\tau) d\tau$

$$\Longrightarrow \int_{t_n}^{t_{n+1}} \ddot{\mathbf{q}}(\tau)d\tau = \int_{t_n}^{t_{n+1}} \left((1-\gamma)\ddot{\mathbf{q}}_n + \gamma\ddot{\mathbf{q}}_{n+1} + \mathbf{q}^{(3)}(\tau)(\tau - h\gamma - t_n) + \mathcal{O}(h^2\mathbf{q}^{(4)})\right) d\tau$$

$$= (1-\gamma)h\,\ddot{\mathbf{q}}_n + \gamma h\,\ddot{\mathbf{q}}_{n+1} + \int_{t_n}^{t_{n+1}} \mathbf{q}^{(3)}(\tau)(\tau - h\gamma - t_n)d\tau + \mathcal{O}(h^3\mathbf{q}^{(4)})$$

- Apply the mean value theorem

$$\Longrightarrow \int_{t_n}^{t_{n+1}} \ddot{\mathbf{q}}(\tau)d\tau = (1-\gamma)h\,\ddot{\mathbf{q}}_n + \gamma h\,\ddot{\mathbf{q}}_{n+1} + \mathbf{q}^{(3)}(\tilde{\tau}) \left[\frac{(\tau - h\gamma - t_n)^2}{2}\right]_{t_n}^{t_{n+1}} + \mathcal{O}(h^3\mathbf{q}^{(4)})$$

$$= (1-\gamma)h\,\ddot{\mathbf{q}}_n + \gamma h\,\ddot{\mathbf{q}}_{n+1} + (\frac{1}{2} - \gamma)h^2\mathbf{q}^{(3)}(\tilde{\tau}) + \mathcal{O}(h^3\mathbf{q}^{(4)})$$

- Substitute the 2^{nd} expression of $\ddot{\mathbf{q}}(\tau)$ in $\int_{t_n}^{t_{n+1}} \ddot{\mathbf{q}}(\tau)(t_{n+1} - \tau)d\tau$

$$\Longrightarrow \int_{t_n}^{t_{n+1}} \ddot{\mathbf{q}}(\tau)(t_{n+1} - \tau)d\tau = (\frac{1}{2} - \beta)h^2\ddot{\mathbf{q}}_n + \beta h^2\ddot{\mathbf{q}}_{n+1} + (\frac{1}{6} - \beta)h^3\mathbf{q}^{(3)}(\tilde{\tau}) + \mathcal{O}(h^4\mathbf{q}^{(4)})$$

- In summary
  - $\forall\gamma$ and $\forall\beta$, the following holds true

$$\int_{t_n}^{t_{n+1}} \ddot{\mathbf{q}}(\tau)d\tau = (1-\gamma)h\,\ddot{\mathbf{q}}_n + \gamma h\,\ddot{\mathbf{q}}_{n+1} + \mathbf{r}_n$$

$$\int_{t_n}^{t_{n+1}} \ddot{\mathbf{q}}(\tau)(t_{n+1}-\tau)d\tau = \left(\frac{1}{2}-\beta\right)h^2\ddot{\mathbf{q}}_n + \beta h^2\ddot{\mathbf{q}}_{n+1} + \mathbf{r}'_n$$

    where

$$\mathbf{r}_n = \left(\frac{1}{2}-\gamma\right)h^2\mathbf{q}^{(3)}(\tilde{\tau}) + \mathcal{O}(h^3\mathbf{q}^{(4)}); \quad \mathbf{r}'_n = \left(\frac{1}{6}-\beta\right)h^3\mathbf{q}^{(3)}(\tilde{\tau}) + \mathcal{O}(h^4\mathbf{q}^{(4)})$$

    and $t_n < \tilde{\tau} < t_{n+1}$
  - neglecting each of $\mathbf{r}_n$ and $\mathbf{r}'_n$ on the ground that they are higher-order functions of the time-step h leads to the following family of time-integration schemes (Newmark's family) for solving (1)

$$\dot{\mathbf{q}}_{n+1} = \dot{\mathbf{q}}_n + (1-\gamma)h\,\ddot{\mathbf{q}}_n + \gamma h\,\ddot{\mathbf{q}}_{n+1} \quad (5)$$

$$\mathbf{q}_{n+1} = \mathbf{q}_n + h\,\dot{\mathbf{q}}_n + h^2\left(\frac{1}{2}-\beta\right)\ddot{\mathbf{q}}_n + h^2\beta\ddot{\mathbf{q}}_{n+1} \quad (6)$$

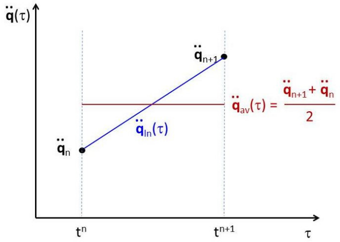
    where γ and β are quadrature parameters

- Particular values of the parameters γ and β
  - $\gamma = \dfrac{1}{2}$ and $\beta = \dfrac{1}{6}$ corresponds to linearly interpolating $\ddot{\mathbf{q}}(\tau)$ in $[t_n, t_{n+1}]$

$$\bar{\ddot{\mathbf{q}}}_{ln}(\tau) = \ddot{\mathbf{q}}_n + (\tau - t_n)\left(\frac{\ddot{\mathbf{q}}_{n+1} - \ddot{\mathbf{q}}_n}{h}\right)$$

  - $\gamma = \dfrac{1}{2}$ and $\beta = \dfrac{1}{4}$ corresponds to averaging $\ddot{\mathbf{q}}(\tau)$ in $[t_n, t_{n+1}]$

$$\bar{\ddot{\mathbf{q}}}_{av}(\tau) = \frac{\ddot{\mathbf{q}}_{n+1} + \ddot{\mathbf{q}}_n}{2}$$



- Application to the direct time-integration of $\mathbf{M}\ddot{\mathbf{q}} + \mathbf{C}\dot{\mathbf{q}} + \mathbf{K}\mathbf{q} = \mathbf{p}(t)$
  - write the equilibrium equation at t_{n+1} and substitute the expressions (5) and (6) into it

$$\Longrightarrow [\mathbf{M} + \gamma h\mathbf{C} + \beta h^2\mathbf{K}]\ddot{\mathbf{q}}_{n+1} = \mathbf{p}_{n+1} - \mathbf{C}[\dot{\mathbf{q}}_n + (1-\gamma)h\ddot{\mathbf{q}}_n] - \mathbf{K}\left[\mathbf{q}_n + h\dot{\mathbf{q}}_n + \left(\frac{1}{2} - \beta\right) h^2\ddot{\mathbf{q}}_n\right]$$

  - if the time-step h is uniform, $\mathbf{M} + \gamma h\mathbf{C} + \beta h^2\mathbf{K}$ can be factored once
  - solve the above system of equations for $\ddot{\mathbf{q}}_{n+1}$
  - substitute the result into the expressions (5) and (6) to obtain $\dot{\mathbf{q}}_{n+1}$ and $\mathbf{q}_{n+1}$

- A time-integration scheme is said to be <u>consistent</u> if

$$\lim_{h \to 0} \frac{\mathbf{u}_{n+1} - \mathbf{u}_n}{h} = \dot{\mathbf{u}}(t_n)$$

- The Newmark time-integration method is consistent

$$\lim_{h \to 0} \frac{\mathbf{u}_{n+1} - \mathbf{u}_n}{h} = \lim_{h \to 0} \begin{bmatrix} (1-\gamma)\ddot{\mathbf{q}}_n + \gamma\ddot{\mathbf{q}}_{n+1} \\ \dot{\mathbf{q}}_n + \left(\frac{1}{2} - \beta\right) h\,\ddot{\mathbf{q}}_n + \beta h\,\ddot{\mathbf{q}}_{n+1} \end{bmatrix} = \begin{bmatrix} \ddot{\mathbf{q}}_n \\ \dot{\mathbf{q}}_n \end{bmatrix}$$

- Consistency is one necessary condition for convergence

- A time-integration scheme is said to be stable if there exists an integration time-step $h_0 > 0$ so that for any $h \in [0, h_0]$, a finite variation of the state vector at time t_n induces only a non-increasing variation of the state-vector $\mathbf{u}_{n+j}$ calculated at a subsequent time t_{n+j}
- Stability is the other necessary condition for convergence

- Premultiplying Eq. (5) and Eq. (6) by **M** and taking into account the equations of equilibrium (1) at t_n and t_{n+1} leads after some algebraic manipulations to

$$
\begin{aligned}
\mathbf{M}\dot{\mathbf{q}}_{n+1} &= \mathbf{M}\dot{\mathbf{q}}_n + h(1-\gamma)[-\mathbf{C}\dot{\mathbf{q}}_n - \mathbf{K}\mathbf{q}_n + \mathbf{p}_n] \\
&+ \gamma h[-\mathbf{C}\dot{\mathbf{q}}_{n+1} - \mathbf{K}\mathbf{q}_{n+1} + \mathbf{p}_{n+1}] \\
\mathbf{M}\mathbf{q}_{n+1} &= \mathbf{M}\mathbf{q}_n + h\mathbf{M}\dot{\mathbf{q}}_n + (\frac{1}{2}-\beta)h^2[-\mathbf{C}\dot{\mathbf{q}}_n - \mathbf{K}\mathbf{q}_n + \mathbf{p}_n] \\
&+ \beta h^2[-\mathbf{C}\dot{\mathbf{q}}_{n+1} - \mathbf{K}\mathbf{q}_{n+1} + \mathbf{p}_{n+1}] \qquad (7)
\end{aligned}
$$

- Equations (7) can be re-written in matrix form as

$$\mathbf{u}_{n+1} = \mathbf{A}(h)\mathbf{u}_n + \mathbf{g}_{n+1}(h)$$

where $\mathbf{A}$ is the amplification matrix associated with the integration operator

$$\mathbf{A}(h) = \mathbf{H}_1^{-1}(h)\mathbf{H}_0(h), \quad \mathbf{g}_{n+1} = \mathbf{H}_1^{-1}(h)\mathbf{b}_{n+1}(h)$$

$$\mathbf{b}_{n+1} = \begin{bmatrix} (1-\gamma)h\mathbf{p}_n + \gamma h\mathbf{p}_{n+1} \\ \left(\frac{1}{2}-\beta\right)h^2\mathbf{p}_n + \beta h^2\mathbf{p}_{n+1} \end{bmatrix}, \quad \mathbf{H}_1 = \begin{bmatrix} \mathbf{M}+\gamma h\mathbf{C} & \gamma h\mathbf{K} \\ \beta h^2\mathbf{C} & \mathbf{M}+\beta h^2\mathbf{K} \end{bmatrix}$$

$$\mathbf{H}_0 = -\begin{bmatrix} -\mathbf{M}+(1-\gamma)h\mathbf{C} & (1-\gamma)h\mathbf{K} \\ \left(\frac{1}{2}-\beta\right)h^2\mathbf{C} - h\mathbf{M} & -\mathbf{M}+\left(\frac{1}{2}-\beta\right)h^2\mathbf{K} \end{bmatrix}$$

- Effect of an initial disturbance
  - $\delta \mathbf{u}_0 = \mathbf{u}_0' - \mathbf{u}_0$

$$\Longrightarrow \delta \mathbf{u}_{n+1} = \mathbf{A}(h)\delta \mathbf{u}_n = \mathbf{A}^2(h)\delta \mathbf{u}_{n-1} = \cdots = \mathbf{A}(h)^{n+1}\delta \mathbf{u}_0$$

  - consider the eigenpairs of $\mathbf{A}(h)$

$$(\lambda_r, \ \mathbf{x}_r)$$

  - then

$$\delta \mathbf{u}_{n+1} = \mathbf{A}^{n+1}(h)\sum_{s=1}^{2N} a_s \mathbf{x}_s = \sum_{s=1}^{2N} a_s \lambda_s^{n+1} \mathbf{x}_s$$

    where N is the dimension of the semi-discrete second-order dynamical system
    $\Longrightarrow \delta \mathbf{u}_{n+1}$ will be amplified by the time-integration operator only if the modulus of an eigenvalue of $\mathbf{A}(h)$ is greater than unity
    $\Longrightarrow \delta \mathbf{u}_{n+1}$ will not be amplified by the time-integration operator if all moduli of all eigenvalues of $\mathbf{A}(h)$ are less than unity

- Undamped case
  - decouple the equations of equilibrium by writing them (for the purpose of analysis) in the modal basis

$$\mathbf{q} = \mathbf{Q}\mathbf{y} = \sum_{i=1}^{N} y_i \mathbf{q}_{a_i} \Longrightarrow \ddot{y}_i + \omega_i^2 y_i = p_i(t)$$

  - apply the Newmark scheme to the i-th modal equation recalled above to obtain the amplification matrix

$$\mathbf{A}(h) = \begin{bmatrix} 1 - \gamma \frac{\omega_i^2 h^2}{1+\beta\omega_i^2 h^2} & -\omega_i^2 h^2 \left(1 - \frac{\gamma}{2} \frac{\omega_i^2 h^2}{1+\beta\omega_i^2 h^2}\right) \\ \frac{h}{1+\beta\omega_i^2 h^2} & 1 - \frac{1}{2} \frac{\omega_i^2 h^2}{1+\beta\omega_i^2 h^2} \end{bmatrix}$$

  - characteristic equation is $\lambda^2 - \lambda\left(2 - (\gamma + \frac{1}{2})\eta^2\right) + 1 - (\gamma - \frac{1}{2})\eta^2 = 0$ where

$$\eta^2 = \frac{\omega_i^2 h^2}{1 + \beta\omega_i^2 h^2}$$

  - characteristic equation has:
    - a pair of complex conjugate roots λ_1 and λ_2 if $\left(\gamma + \frac{1}{2}\right)^2 - 4\beta \leq \frac{4}{\omega_i^2 h^2} \Leftrightarrow \left(\gamma + \frac{1}{2}\right)^2 \eta^2 < 4, \quad i = 1, \cdots, N$ (case 1)
    - two identical real roots if $\left(\gamma + \frac{1}{2}\right)^2 \eta^2 = 4$ (case 2)
    - two distinct real roots if $\left(\gamma + \frac{1}{2}\right)^2 \eta^2 > 4$ (case 3)

- Undamped case (continue)
  - it can be shown that case 1 is the limiting case, in which case

$$\lambda_{1,2} = \rho e^{\pm i\phi}$$

where

$$\rho = \sqrt{1 - \left(\gamma - \frac{1}{2}\right)\eta^2}$$

$$\phi = \arctan\left(\frac{\eta\sqrt{1 - \frac{1}{4}(\gamma + \frac{1}{2})^2\eta^2}}{1 - \frac{1}{2}(\gamma + \frac{1}{2})\eta^2}\right)$$

  - then, the Newmark scheme is stable if

$$\rho \leq 1 \Rightarrow \gamma \geq \frac{1}{2}$$

and

$$\left(\gamma + \frac{1}{2}\right)^2 - 4\beta \leq \frac{4}{\omega_i^2 h^2}, \quad i = 1, \cdots, N$$

$\Longrightarrow$ limitation on the maximum time-step

- Undamped case (continue)
  - the algorithm is *conditionally* stable if

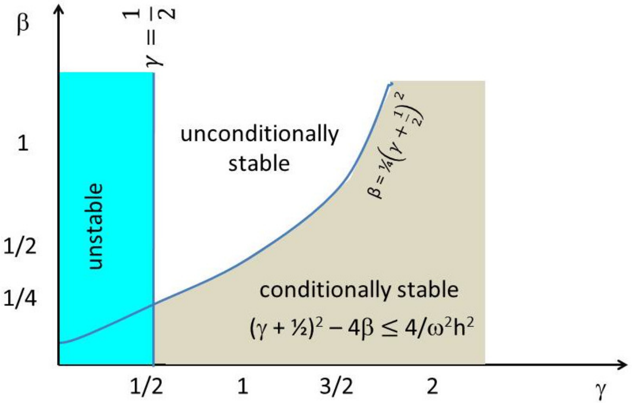
$$\boxed{\gamma \geq \frac{1}{2}}$$

  - it is *unconditionally* stable if furthermore $\beta \geq \frac{1}{4}\left(\gamma + \frac{1}{2}\right)^2$ — that is,

$$\boxed{\gamma \geq \frac{1}{2} \quad \text{and} \quad \beta \geq \frac{1}{4}\left(\gamma + \frac{1}{2}\right)^2}$$

  - the choice $\gamma = \frac{1}{2}$ and $\beta = \frac{1}{4}$ leads to an unconditionally stable time-integration operator of maximum accuracy

- Undamped case (continue)



Stability of the Newmark scheme

- Damped case ($\mathbf{C} \neq \mathbf{0}$)
  - consider the case of modal damping
  - then, the uncoupled equations of motion are

$$\ddot{y}_i + 2\xi_i \omega_i \dot{y}_i + \omega_i^2 y_i = p_i(t)$$

    where ξ_i is the modal damping coefficient
  - consider the case $\gamma = \dfrac{1}{2}, \ \beta = \dfrac{1}{4}$
  - an analysis similar to that performed in the undamped case reveals that in this case, the Newmark scheme remains stable as long as $\xi_i < 1$
  - in general, damping has a stabilizing effect for moderate values of ξ_i

- Free-vibration of an undamped linear oscillator

$$\ddot{y} + \omega^2 y = 0 \qquad \text{and} \qquad y(0) = y_0,\ \dot{y}(0) = 0 \qquad \mathbf{A} = \begin{bmatrix} 0 & -\omega_0^2 \\ 1 & 0 \end{bmatrix}$$

  - the above problem has an exact solution $y(t) = y_0 \cos \omega t$ which can be written in complex discrete form as $y_{n+1} = e^{i\omega h} y_n \Rightarrow$ the exact amplification factor is $\rho_{ex} = 1$ and the exact phase is $\phi_{ex} = \omega h$
  - the numerical solution satisfies

$$\mathbf{u}_{n+1} = \begin{bmatrix} \dot{y}_{n+1} \\ y_{n+1} \end{bmatrix} = \mathbf{A}(h)\mathbf{u}_n$$

  - let $\lambda_{1,2}(\beta, \gamma)$ be the eigenvalues of $\mathbf{A}(h, \beta, \gamma)$
  - when $\left(\gamma + \frac{1}{2}\right)^2 - 4\beta \leq \frac{4}{\omega_i^2 h^2}$, λ_1 and λ_2 are complex-conjugate

$$\lambda_{1,2}(\beta, \gamma) = \rho(\beta, \gamma) e^{\pm i\phi(\beta,\gamma)}$$

where

$$\rho = \sqrt{1 - \left(\gamma - \frac{1}{2}\right)\eta^2}, \quad \phi = \arctan\left(\frac{\eta\sqrt{1 - \frac{1}{4}(\gamma + \frac{1}{2})^2\eta^2}}{1 - \frac{1}{2}(\gamma + \frac{1}{2})\eta^2}\right), \quad \eta^2 = \frac{\omega^2 h^2}{1 + \beta\omega^2 h^2}$$

- Free-vibration of an undamped linear oscillator (continue)
  - amplitude error

$$\rho - \rho_{ex} = \rho - 1 = -\frac{1}{2}\left(\gamma - \frac{1}{2}\right)\omega^2 h^2 + \mathcal{O}(h^4)$$

  - relative periodicity error

$$\frac{\Delta T}{T} = \frac{\Delta \frac{1}{\phi}}{\frac{1}{\phi}} = \frac{\frac{1}{\phi} - \frac{1}{\phi_{ex}}}{\frac{1}{\phi_{ex}}} = \frac{\omega h}{\phi} - 1 = \frac{1}{2}\left(\beta - \frac{1}{12}\right)\omega^2 h^2 + \mathcal{O}(h^3)$$

Algorithm	γ	β	Stability limit ωh	Amplitude error $\rho - 1$	Periodicity error $\frac{\Delta T}{T}$
Purely explicit	0	0	0	$\frac{\omega^2 h^2}{4}$	—
Central difference	$\frac{1}{2}$	0	2	0	$-\frac{\omega^2 h^2}{24}$
Fox & Goodwin	$\frac{1}{2}$	$\frac{1}{12}$	2.45	0	$\mathcal{O}(h^3)$
Linear acceleration	$\frac{1}{2}$	$\frac{1}{6}$	3.46	0	$\frac{\omega^2 h^2}{24}$
Average constant acceleration	$\frac{1}{2}$	$\frac{1}{4}$	∞	0	$\frac{\omega^2 h^2}{12}$

Table: Time-integration schemes of the Newmark family

- The purely explicit scheme ($\gamma = 0,\ \beta = 0$) is useless
- The Fox & Godwin scheme has asymptotically the smallest phase error but is only conditionally stable
- The average constant acceleration scheme ($\gamma = \dfrac{1}{2}, \beta = \dfrac{1}{4}$) is the unconditionally stable scheme with asymptotically the highest accuracy

- Conservation of total energy
  - dynamic system with scleronomic constraints

$$\frac{d}{dt}(\mathcal{T} + \mathcal{V}) = -m\mathcal{D} + \sum_{s=1}^{n_s} Q_s \dot{q}_s$$

  - $\mathcal{T} = \frac{1}{2}\dot{\mathbf{q}}^T\mathbf{M}\dot{\mathbf{q}}$ and $\mathcal{V} = \frac{1}{2}\mathbf{q}^T\mathbf{K}\mathbf{q}$
  - the dissipation function $\mathcal{D}$ is a quadratic function of the velocities ($m = 2$)

$$\mathcal{D} = \frac{1}{2}\dot{\mathbf{q}}^T\mathbf{C}\dot{\mathbf{q}}$$

  - external force component of the power balance

$$\sum_{s=1}^{n_s} Q_s \dot{q}_s = \dot{\mathbf{q}}^T\mathbf{p}$$

  - integration over a time-step $[t_n, t_{n+1}]$

$$[\mathcal{T} + \mathcal{V}]_{t_n}^{t_{n+1}} = \int_{t_n}^{t_{n+1}} (-\dot{\mathbf{q}}^T\mathbf{C}\dot{\mathbf{q}} + \dot{\mathbf{q}}^T\mathbf{p})dt$$

- Conservation of total energy (continue)
  - note that because $\mathbf{M}$ and $\mathbf{K}$ are symmetric ($\mathbf{M}^T = \mathbf{M}$ and $\mathbf{K}^T = \mathbf{K}$)

$$
\begin{aligned}
[\mathcal{T} + \mathcal{V}]_{t_n}^{t_{n+1}} = [\mathcal{T}_{n+1} - \mathcal{T}_n] + [\mathcal{V}_{n+1} - \mathcal{V}_n] &= \frac{1}{2}(\dot{\mathbf{q}}_{n+1} - \dot{\mathbf{q}}_n)^T \mathbf{M}(\dot{\mathbf{q}}_{n+1} + \dot{\mathbf{q}}_n) \\
&+ \frac{1}{2}(\mathbf{q}_{n+1} - \mathbf{q}_n)^T \mathbf{K}(\mathbf{q}_{n+1} + \mathbf{q}_n)
\end{aligned}
$$

  - when time-integration is performed using the Newmark algorithm with $\gamma = \frac{1}{2}, \ \beta = \frac{1}{4}$, the above variation becomes (see (5) and (6))

$$
\boxed{[\mathcal{T} + \mathcal{V}]_{t_n}^{t_{n+1}} = \frac{1}{2}(\mathbf{q}_{n+1} - \mathbf{q}_n)^T (\mathbf{p}_n + \mathbf{p}_{n+1}) - \frac{h}{4}(\dot{\mathbf{q}}_{n+1} + \dot{\mathbf{q}}_n)^T \mathbf{C}(\dot{\mathbf{q}}_{n+1} + \dot{\mathbf{q}}_n)}
$$

  - when applied to a conservative system ($\mathbf{C} = \mathbf{0}$ and $\mathbf{p} = \mathbf{0}$), preserves the total energy
  - for non-conservative systems, $[\mathcal{T} + \mathcal{V}]_{t_n}^{t_{n+1}} = \int_{t_n}^{t_{n+1}} (-\dot{\mathbf{q}}^T \mathbf{C}\dot{\mathbf{q}} + \dot{\mathbf{q}}^T \mathbf{p}) dt$ and therefore both terms in the right-hand side of the above formula result from numerical quadrature relationships that are consistent with the time-integration operator

$$
\begin{aligned}
\int_{t_n}^{t_{n+1}} \dot{\mathbf{q}}^T \mathbf{p} dt &\approx \left(\int_{t_n}^{t_{n+1}} \dot{\mathbf{q}}^T dt\right) \left(\frac{\mathbf{p}_n + \mathbf{p}_{n+1}}{2}\right) = \frac{1}{2}(\mathbf{q}_{n+1} - \mathbf{q}_n)^T (\mathbf{p}_n + \mathbf{p}_{n+1}) \\
\int_{t_n}^{t_{n+1}} \dot{\mathbf{q}}^T \mathbf{C}\dot{\mathbf{q}} dt &\approx \left(\int_{t_n}^{t_{n+1}} \dot{\mathbf{q}}^T dt\right) \mathbf{C} \left(\frac{\dot{\mathbf{q}}_n + \dot{\mathbf{q}}_{n+1}}{2}\right) = \frac{1}{2}(\mathbf{q}_{n+1} - \mathbf{q}_n)^T \mathbf{C} \left(\frac{\dot{\mathbf{q}}_n + \dot{\mathbf{q}}_{n+1}}{2}\right) \\
&= \frac{h}{4}(\dot{\mathbf{q}}_{n+1} + \dot{\mathbf{q}}_n)^T \mathbf{C}(\dot{\mathbf{q}}_{n+1} + \dot{\mathbf{q}}_n)
\end{aligned}
$$

- Central Difference (CD) scheme = Newmark's with $\gamma = \frac{1}{2}$, $\beta = 0$

$$\begin{aligned} \dot{\mathbf{q}}_{n+1} &= \dot{\mathbf{q}}_n + h_{n+1}\left(\frac{\ddot{\mathbf{q}}_n + \ddot{\mathbf{q}}_{n+1}}{2}\right) \\ \mathbf{q}_{n+1} &= \mathbf{q}_n + h_{n+1}\dot{\mathbf{q}}_n + \frac{h_{n+1}^2}{2}\ddot{\mathbf{q}}_n \end{aligned} \tag{8}$$

  where $h_{n+1} = t_{n+1} - t_n$
- Equivalent three-step form
  - start with

$$\mathbf{q}_n = \mathbf{q}_{n-1} + h_n\dot{\mathbf{q}}_{n-1} + \frac{h_n^2}{2}\ddot{\mathbf{q}}_{n-1} = \mathbf{q}_{n-1} + h_n \underbrace{\left(\dot{\mathbf{q}}_{n-1} + \frac{h_n}{2}\ddot{\mathbf{q}}_{n-1}\right)}_{\dot{\mathbf{q}}_{n-\frac{1}{2}}} \tag{9}$$

  - divide by h_n and subtract the result from $\mathbf{q}_{n+1}$ divided by h_{n+1}
  - account for the relationship (8)

$$\Longrightarrow \ddot{\mathbf{q}}_n = \frac{h_n(\mathbf{q}_{n+1} - \mathbf{q}_n) - h_{n+1}(\mathbf{q}_n - \mathbf{q}_{n-1})}{h_{n+\frac{1}{2}} h_n h_{n+1}} \tag{10}$$

  where $h_{n+\frac{1}{2}} = \dfrac{h_n + h_{n+1}}{2}$

- Case of a constant time-step h

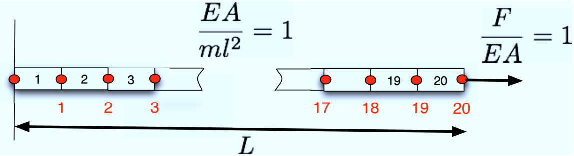
$$\boxed{\ddot{\mathbf{q}}_n = \frac{\mathbf{q}_{n+1} - 2\mathbf{q}_n + \mathbf{q}_{n-1}}{h^2}}$$

- Efficient implementation
  - <u>use a lumped mass matrix $\mathbf{M}$</u>
  - initialize: $\ddot{\mathbf{q}}_0 = \mathbf{M}^{-1}(\mathbf{p}_0 - \mathbf{K}\mathbf{q}_0)$ and $\dot{\mathbf{q}}_{\frac{1}{2}} = \dot{\mathbf{q}}_0 + \frac{h_1}{2}\ddot{\mathbf{q}}_0$
  - increment the displacement: $\mathbf{q}_n = \mathbf{q}_{n-1} + h_n \dot{\mathbf{q}}_{n-\frac{1}{2}}$ (see (9))
  - compute the acceleration: $\ddot{\mathbf{q}}_n = \mathbf{M}^{-1}(\mathbf{p}_n - \mathbf{K}\mathbf{q}_n)$ (enforce equilibrium at t_n)
  - increment the velocity at half time-step (formula results from (10))

$$\dot{\mathbf{q}}_{n+\frac{1}{2}} = \dot{\mathbf{q}}_{n-\frac{1}{2}} + h_{n+\frac{1}{2}} \ddot{\mathbf{q}}_n \Leftrightarrow \ddot{\mathbf{q}}_n = \frac{\dot{\mathbf{q}}_{n+\frac{1}{2}} - \dot{\mathbf{q}}_{n-\frac{1}{2}}}{h_{n+\frac{1}{2}}}$$

- Stability condition: for $\gamma = 1/2$ and $\beta = 0$, $\left(\gamma + \frac{1}{2}\right)^2 - 4\beta \leq \frac{4}{\omega_{cr}^2 h^2} \Rightarrow \omega_{cr} h \leq 2$ where ω_{cr} is the highest frequency contained in the model – this condition is also known as the *Courant condition*
- $h_{cr} = \dfrac{2}{\omega_{cr}}$ is referred to here as the maximum Courant stability time-step

- Clamped bar subjected to a step load at its free end
- Model made of $N = 20$ finite elements with equal length $l = \dfrac{L}{N}$



  - lumped mass matrix
- Eigenfrequencies of the continuous system

$$\omega_{cont_r} = (2r-1)\frac{\pi}{2}\sqrt{\frac{EA}{mL^2}} = \left(\frac{2r-1}{N}\right)\frac{\pi}{2}\sqrt{\frac{EA}{ml^2}} = \left(\frac{2r-1}{N}\right)\frac{\pi}{2}$$

- Finite element stiffness and mass matrices

$$\mathbf{M} = \frac{ml}{2}\begin{bmatrix} 2 & & & & & \\ & 2 & & & \mathbf{0} & \\ & & 2 & & & \\ & & & \ddots & & \\ & \mathbf{0} & & & 2 & \\ & & & & & 1 \end{bmatrix} \quad \mathbf{K} = \frac{EA}{l}\begin{bmatrix} 2 & -1 & & & & \\ -1 & 2 & -1 & & \mathbf{0} & \\ & -1 & 2 & \ddots & & \\ & & \ddots & \ddots & -1 & \\ & \mathbf{0} & & -1 & 2 & -1 \\ & & & & -1 & 1 \end{bmatrix} \tag{11}$$

- Analytical frequencies of the discrete system

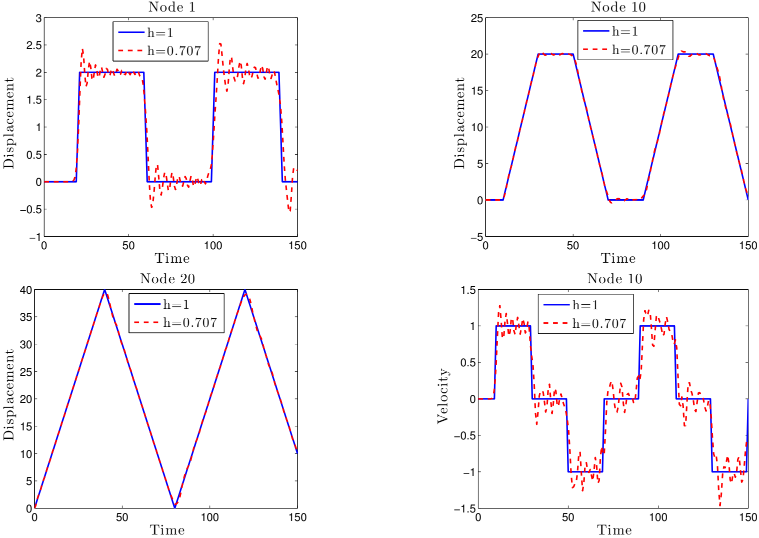
$$\omega_r = 2\sqrt{\frac{EA}{ml^2}}\sin\left(\left(\frac{2r-1}{2N}\right)\frac{\pi}{2}\right) = 2\sin\left(\left(\frac{2r-1}{2N}\right)\frac{\pi}{2}\right), \quad r = 1, 2, \cdots N$$

$$\Rightarrow \quad \omega_{cr} < \omega_{cr}(r = N, N \to \infty) = 2$$

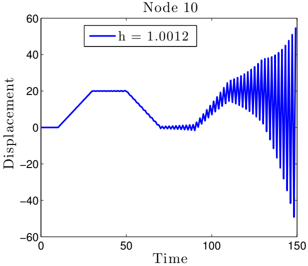
- Critical time-step for the CD algorithm

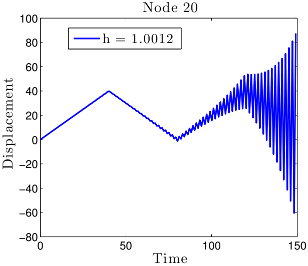
$$\omega_{cr} h_{cr} = 2 \Rightarrow h_{cr} = 1$$

- $h = 1,\ h = 0.707$


Node 1
h=1
h=0.707
Displacement
Time
Node 10
h=1
h=0.707
Displacement
Time
Node 20
h=1
h=0.707
Displacement
Time
Node 10
h=1
h=0.707
Velocity
Time

- $h = 1.0012$


Node 10
h = 1.0012
Displacement
Time
60
40
20
0
−20
−40
−60
0
50
100
150


Node 20
h = 1.0012
Displacement
Time
100
80
60
40
20
0
−20
−40
−60
−80
0
50
100
150

- For the clamped-free bar example, the CD method computes the exact solution when $h = h_{cr}$
- Comparison of the exact solution of the continuous free-vibration bar problem and the analytical expression of the numerical solution
  - denote by $q_{j,n}$ the value of the j-th d.o.f. at time t_n
  - if $q_{j,n}$ is not located at the boundary, it satisfies $\big($see (11)$\big)$

$$\frac{ml}{h^2}(q_{j,n+1} - 2q_{j,n} + q_{j,n-1}) + \frac{EA}{l}(-q_{j-1,n} + 2q_{j,n} - q_{j+1,n}) = 0$$

  - the general solution of the above problem is

$$q_{j,n} = \underbrace{\sin(j\mu + \phi)}_{spatial\ \ component} \underbrace{[a \cos n\theta + b \sin n\theta]}_{temporal\ \ component} \tag{12}$$

  - comparing the above expression to the exact harmonic solution of the continuous form of this free-vibration problem (which can be derived analytically)

$$\Longrightarrow n\theta = \omega t = \omega n h \Rightarrow \frac{\theta}{h} = \omega_{num}$$

- Comparison of the exact solution of the free-vibration bar problem and the analytical expression of the numerical solution (continue)
  - introduce the exact expression for $q_{j,n}$ in the CD scheme

$$2[(1-\cos\mu) - \lambda^2(1-\cos\theta)]q_{j,n} = 0$$

    where $\lambda^2 = \left(\dfrac{ml^2}{EA}\right)\dfrac{1}{h^2} = \dfrac{1}{h^2} \Rightarrow 1-\cos\theta = \dfrac{1}{\lambda^2}(1-\cos\mu)$

  - make use of the boundary conditions in space $\big(q_{0,n} = 0$, and plug (12) in the last equation in (11)$\big)$
    $\Longrightarrow \phi = 0$ and $\mu_r = \left(\frac{2r-1}{N}\right)\frac{\pi}{2}, \quad r \in \mathbb{N}^*$

$$\Longrightarrow 1-\cos\theta_r = \frac{1}{\lambda^2}(1-\cos\mu_r)$$

  - special case $\lambda^2 = 1$ $(h = h_{cr} = 1) \Rightarrow \theta_r = \mu_r$ and

$$\omega_{num_r} = \frac{\theta_r}{h} = \mu_r = \left(\frac{2r-1}{N}\right)\frac{\pi}{2}\sqrt{\frac{EA}{ml^2}} = \left(\frac{2r-1}{N}\right)\frac{\pi}{2}$$

    $\Longrightarrow$ the r-th numerical frequency coincides with the r-th eigenfrequency of the continuous system