Mon April 20
9.6 wave equation solutions via Fourier and d'Alembert

Announcements:

Warm-up Exercise:

The non Fourier series ("d'Alembert") approach to the wave equation: For

$$u_{tt} = a^2 u_{xx} \qquad -\infty < x < \infty$$
$$u(x, 0) = f(x)$$
$$u_t(x, 0) = g(x)$$

the solution is

$$u(x, t) = \frac{1}{2}\left(f(x - at) + f(x + at)\right) + \frac{1}{2\,a}\int_{x - at}^{x + at} g(s)\,\mathrm{d}s$$

$$= \frac{1}{2}\left(f(x - at) + f(x + at)\right) + \frac{1}{2\,a}\left(G(x + a\,t) - G(x - a\,t)\right).$$

(Here, G is any antiderivative of g). You check these facts, more or less, in problems 13-16 of the text.

You can combine the d'Alembert solution above with even and odd extensions of initial data from the interval $0 < x < L$ to solve the natural IBVP's that one can also solve with Fourier series. In this case you are not required to use Fourier series at all! But we'll use Fourier series to illustrate the connection between the two methods as well as to make movies for the traveling wave solutions. Our first example will be for the free-endpoint (type 2) boundary value problem, using the tent function. Our second example will be an impulse wave with fixed endpoints (type 1).

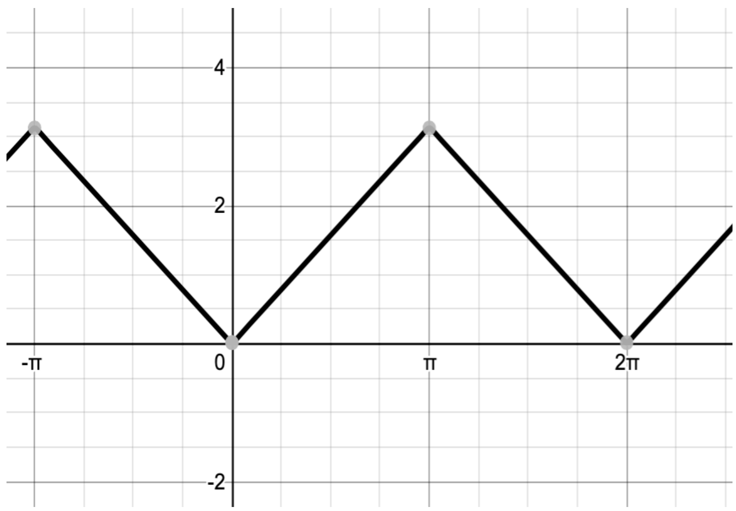
Example 1: Fourier series solution to type 2 boundary condition, re-interpreted in terms of traveling waves.

$$
\begin{aligned}
&u_{tt} = 9\, u_{xx} && 0 < x < 2\pi,\ t > 0\\
&u(x,0) = f(x) && 0 < x < 2\pi\\
&u_t(x,0) = 0 && 0 < x < 2\pi\\
&u_x(0,t) = u_x(L,t) = 0, && t > 0
\end{aligned}
$$

In this example, $f(x)$ is the 2π-periodic $tent(x)$ funtion that $|x|$ from the interval $[-\pi, \pi]$ to $\mathbb{R}$.

$$f(x) = tent(x) = \frac{\pi}{2} - \frac{4}{\pi} \sum_{n = odd} \frac{1}{n^2} \cos(n\,x).$$

Note that this *is* the Fourier series for this $2\,L = 4\,\pi$- periodic function, which also happens to be even and $2\,\pi$ periodic.



1a) Use our building block product solutions and infinite superposition to find a formula for the solution $u(x,t)$

1b) Use addition angle formulas for cos to rewrite the solution as a sum of two traveling waves, one in each direction and with half the profile function as the initial displacement function $f(x) = tent(x)$. We know it has to be true, from page 1, that

$$u(x,t) = \frac{1}{2}\left(tent(x - 3\,t) + tent(x + 3\,t)\right)$$

1c) We worked out the formulas

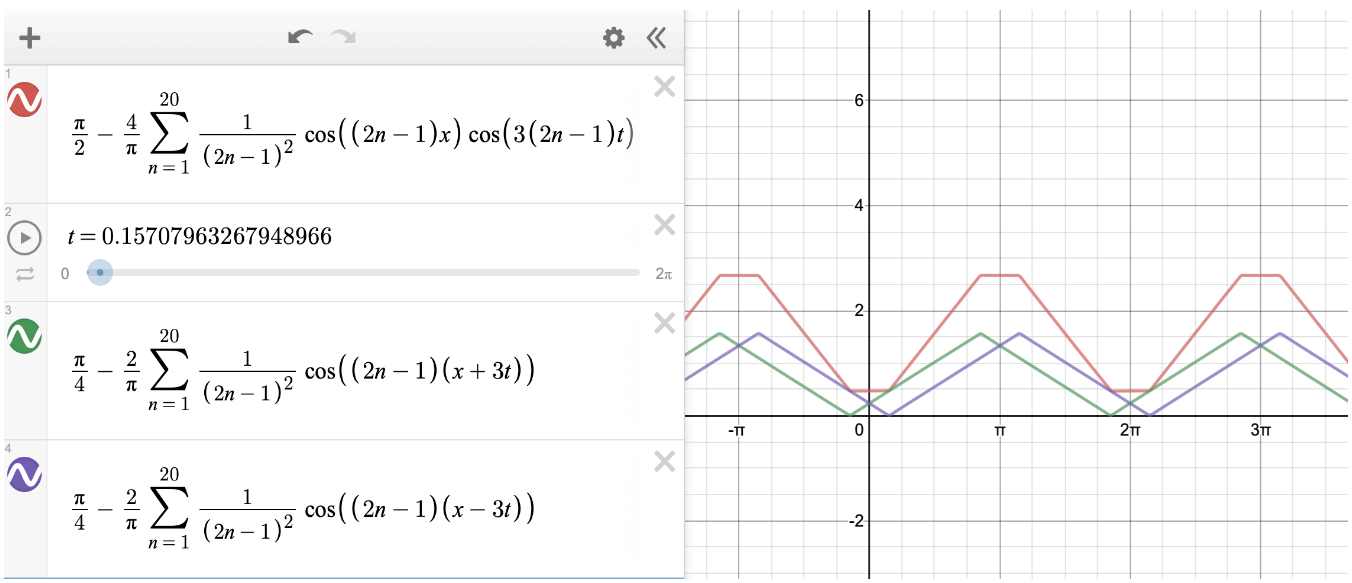
$$u(x,t) = \frac{\pi}{2} - \frac{4}{\pi}\sum_{n=odd}\frac{1}{n^2}\cos(n\,x)\cos(3\,n\,t)$$

$$u(x,t) = \left(\frac{\pi}{4} - \frac{2}{\pi}\sum_{n=odd}\frac{1}{n^2}\cos(n(x-3\,t))\right) + \left(\frac{\pi}{4} - \frac{2}{\pi}\sum_{n=odd}\frac{1}{n^2}\cos(n(x+3\,t))\right)$$

verifying

$$u(x,t) = \frac{1}{2}\left(tent(x-3\,t) + tent(x+3\,t)\right)$$

Now let's watch the movie at Desmos from which this screen shot was taken. The green wave is moving to the left, the purple one is moving to the right, and the red one is their superposition, $u(x,t)$:



Example 2: This is for fun - if you ever take a physics class on wave phenomena you might see stuff like this. If we were still meeting in our classroom, which I miss, on the last day of class I would've brought in a slinky and we would've played with it. This visualization is the best we can do remotely.

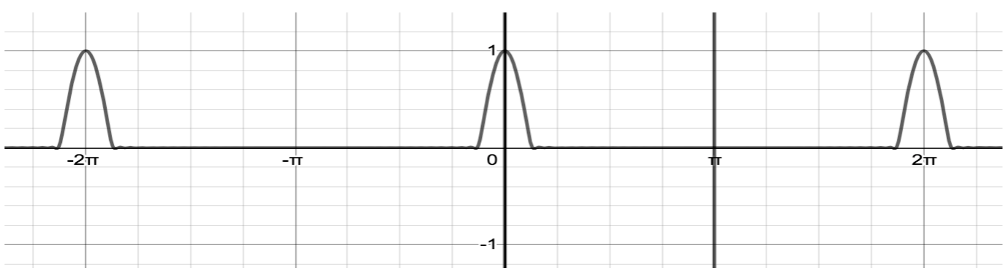
Imagine: Two volunteers would've stretched ithe slinky practically the length of the classroom. We would've demonstrated the fundamental mode, and maybe the second mode of oscillation, for transverse oscillations with fixed endpoints. We would've shown that the slinky makes compression waves as well as transverse waves. By an accident of slinkiness, they each have approximately same wave speed, which we would've verified from our mathematical models. (This turns out to be because the slinky's tension is roughly proportional to the length it's been stretched.) We would've measured the speed of compression vs. transverse waves to test our math.

Finally, we would've shown that with fixed endpoints, a transverse impulse sent down the slinky by a person at one end, comes back the other way upside down. Which seems strange, until you think of superposition. Because you can envision the superposition of a wave train going in one direction with its opposite, going in the opposite direction, in such a way that at the ends of the slinky the two trains always cancel out exactly. Making the endpoints fixed.

With Desmos, we can pretend we're seeing the real thing! Consider the profile function on the interval $[0, \pi]$

$$f(x) = \begin{cases} \cos(4x) & 0 < x < \dfrac{\pi}{8} \\[2ex] 0 & \dfrac{\pi}{8} < x < \pi \end{cases}$$

The even extension $f_{even}(x)$ has this graph, courtesy of Fourier series and Desmos:



I used technology (Maple, in my case) to compute the Fourier coefficients for $f_{even}(t)$. This is the formula that results:

$$f_{even}(x) = \frac{1}{4\pi} + \frac{1}{8}\cos(4x) - \frac{8}{\pi}\sum_{n \neq 4} \frac{\cos\left(\frac{n\pi}{8}\right)}{n^2 - 16}\cos(nx).$$

We will use superposition and traveling waves for the wave equation with fixed boundary. Notice I haven't specified the initial displacement and velocity.

$$u_{tt} = 4\,u_{xx} \qquad 0 < x < \pi, t > 0$$
$$u(0,t) = u(\pi,t) = 0, \qquad t > 0$$

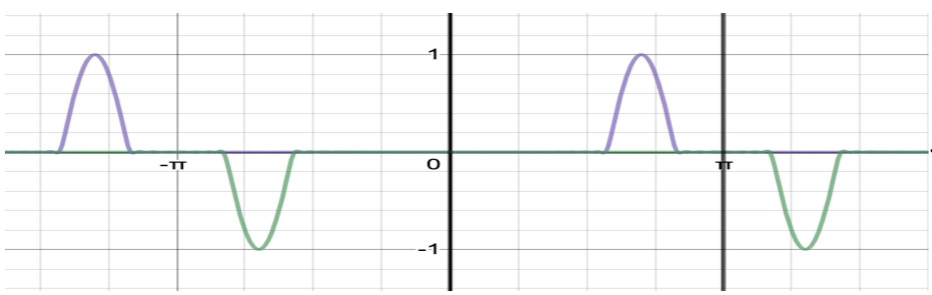
By the way we chose $f_{even}(x)$, the following superposition of wave trains going in opposite directions and with opposite sign will create a type-1 solution to the wave equation on the interval $[0, \pi]$:

$$u(x,t) = f_{even}(x - 2t) - f_{even}(x + 2t)$$

which after addition angle formulas for the sum of the cos series can also be rewritten as

$$u(x,t) = \frac{1}{4}\sin(4x)\sin(8t) - \frac{16}{\pi}\sum_{n \neq 4} \frac{\cos\left(\frac{n\pi}{8}\right)}{n^2 - 16}\sin(nx)\sin(2nt)$$

The purple bumps are moving to the right, the green ones to the left. The sum of the two functions is always zero at the endpoints $x = 0, \pi$:



And this explains why the transverse wave comes back upside down! It's the purple one as it moves to the right, a hybrid when it's near $x = \pi$,
and then the green curve when it moves back to the left. A virtual slinky!

We'll watch a movie at Desmos.

Upshot: Math is fun, especially with visualization.