2.2 The spin states in the Stern-Gerlach experiment are analogous to the behavior of plane and circularly polarized light

1. We will show here that when we consider the behavior of plane and circularly polarized light (which are considered in wave-forms), then we get behavior identical to what was seen from the spin states in the Stern-Gerlach experiments.

2. With this we hope to convince ourselves that the spin states in the Stern-Gerlach experiment are *really acting like waves and like vectors(!!)* thus making all our previous observations meaningful, and hence leading to the basis for the **wave particle duality** arguments later (in c.a. 1927) proposed by de Broglie.

3. **What is plane polarized light?**

   - Light is made of electric and magnetic fields.
   - These are vectors. See the link: http://www.indiana.edu/%7Essiweb/C561/movies/EandManim.gif for an animated rendering of the electric and magnetic fields in light.
   - And light is generally represented by a right handed (three-dimensional) set of vectors.
   - The electric and magnetic fields themselves are waves of the kind:

   $$E = E_0 \hat{x} \cos(kz - \omega t) \tag{2.6}$$

   where z is the direction in which the light waves are moving in time, and k and ω are the wave-vector and the frequency of light.
   - The magnetic field vector of light is along the $\hat{y}$ direction.
   - Note that in the equation above the electric field remains forever on the x-z plane. Such a wave is said to be plane-polarized with the direction of polarization along x-axis.

4. Now consider the following experiment with plane-polarized light.
   - Consider a filter that creates a plane-polarized light in the x-direction.
   - This is the same filter that you guys might have encountered in a general chemistry or P. Chem experiment.
   - The way this filter works is, it allows only plane-polarized light in the x-direction to come out of it. But you may remember that by rotating the knob you could get plane-polarization at different angles.
   - So lets remember that experiment before we move further:
     - There was a polarizer that light was fed into.
     - By rotating the knob you could see the intensity reduce or increase.
     - The following figure depicts this experiment completely.

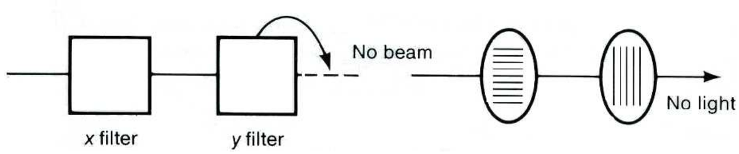


Figure 6: The plane-polarized light experiment

5. Look at this website for some really nice experiments using plane (and circularly) polarized light. This will also allow you to visualize some of the mathematics we will do here.

6. Now consider the following analogy between the Stern-Gerlach SG_z^+ filter and the x-filter. (We have chosen to use the term filter for the SG experiment, because thats what it does, it filters out everything but the state that has a positive z-axis contribution of S_z.)

Figure 7:

7. Does this analogy make sense? The x-filter allows only plane polarized light in the x-direction. Similarly the SG_z^+ filter on the right side of the above figure only allows states with +ve z-axis contribution of S_z. Perhaps the following picture makes it clearer:

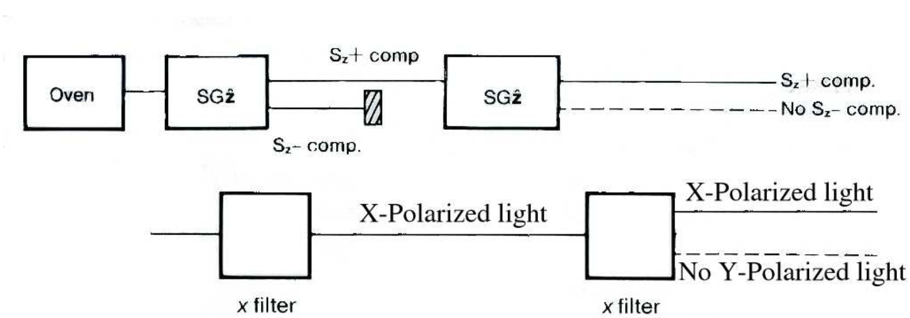
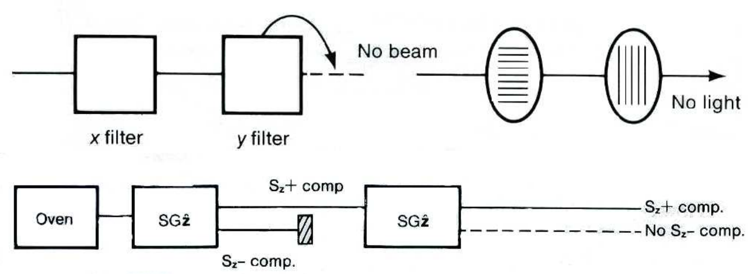


Figure 8:

8. Now consider the following analogy between the Stern-Gerlach SG_z^- filter and the y-filter.

Figure 9:

9. And a similar diagram like Fig. (7) can be drawn for the y-filter.
10. Note further that x-polarized light fed into a y-filter does not yield any light in a fashion similar to the fact that an SG_z^+ filter does not yield any states that have negative z-axis contribution of S_z. (Compare Fig. (6) and Fig. (2) which are both reproduced below for your convenience.)



11. Thus the analogy between the x- and y-filters with the SG_z^+ and SG_z^- filters in now complete.
12. *The Stern Gerlach experimental observations are hence very similar to observations conducted on plane-polarized light.*

13. Now lets consider an analogy for the SG_x^+ and SG_x^- filters.

14. Consider a polarization direction x′ that is 45 degrees rotated from the x-direction. (y′-polarization direction is 90 degrees rotated from x′-direction.)

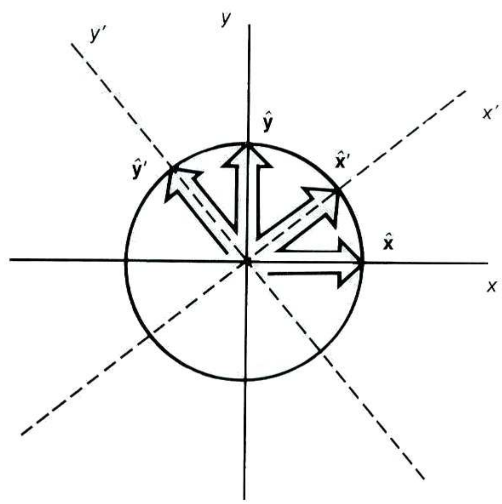


Figure 10: Direction of the x′ is 45 degrees rotated from the x-direction

15. Consider the following polarization sequence.

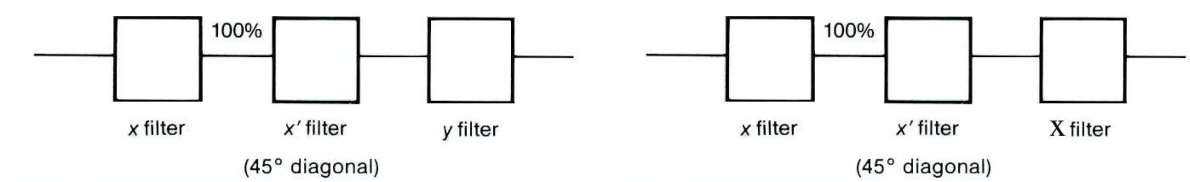
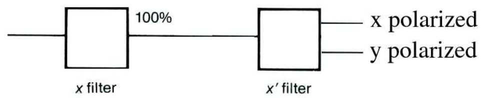
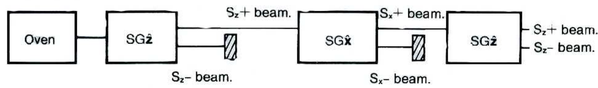


Figure 11: You get light from both. Compare this figure with Fig. (5).

16. Compare Fig. (11) above with the Fig. (5) which are reproduced below for your convenience. It does seem like the x′ filter acts in a fashion similar to the SG_x^+ filter does.





17. Similarly the SG_x^- filter acts like the y′ filter. **Homework: Can you argue this in a similar fashion as what we have done above?**

18. If $SG_x^+ \leftrightarrow$ x′-filter, can we say that $SG_x^- \leftrightarrow$ y′-filter?

19. But:
$$\hat{x'} = \frac{1}{\sqrt{2}} \left[\hat{x} + \hat{y} \right] \tag{2.7}$$
and
$$\hat{y'} = \frac{1}{\sqrt{2}} \left[\hat{x} - \hat{y} \right] \tag{2.8}$$
Due to analogy constructed we must be able to same the same for the Stern-Gerlach states:
$$SG_x^+ = \frac{1}{\sqrt{2}} \left[SG_z^+ + SG_z^- \right] \tag{2.9}$$
and
$$SG_x^- = \frac{1}{\sqrt{2}} \left[SG_z^+ - SG_z^- \right] \tag{2.10}$$

20. Hence we may construct this analogy between polarized light and the states for the Stern-Gerlach experiment.

21. **But wait, we have now resolved the confusion we had in Figure (5). Now it all makes sense when we look at Figure (5) in the same manner as Figure (11).**
22. But this analogy is profound. As we will discuss in class.
23. So it must be that the spin states act in a fashion similar to plane polarized light (which are waves as we saw in Eq. (2.6)).
24. This is precisely what de Broglie was to realize (*the wave particle-duality*) a few years after Stern and Gerlach did this experiment. But now we can see how it all comes together.
25. Another aspect that becomes clear from the analogy to polarized light is that *vectors* have a required property to describe these states. (Note: The electric field directions are vectors. See Eq. (2.6) and also see Figure (14).) For this reason we consider it important to embark into a discussion of vector spaces. The reason behind your math handouts.
26. The x′ polarizer changes an input x-polarized beam to an x′ polarized beam. So, something similar is going on here with respect to the SG_x operation on an S_z^+ input.
27. *This is actually a very subtle concept in quantum mechanics that is completely non-existent in classical theories.* We will see later on that all makes perfect sense when the correct mathematical framework is utilized.

Homework:

Circularly polarized light: In this homework, we will: (a) first present an expression for circularly polarized (Eq. (2.11) below), (b) simplify this expression using an identity from complex numbers, (c) convince ourselves (pictorially) that the expression we presented is indeed circularly polarized, (d) construct an analogy between SG_y^+ with circularly polarized light.

In Equation (2.6) we talked about how plane-polarized light can be represented using a moving wave. But note here that the plane of polarization remains x-z forever. (How do we know that? The electric field remains along the x-axis for all time.) Consider, now, a wave that looks like:

$$E = E_0 \left[\frac{1}{\sqrt{2}} \hat{x} \cos\left(kz - \omega t\right) + \frac{1}{\sqrt{2}} \hat{y} \cos\left(kz - \omega t + \frac{\pi}{2}\right) \right] \tag{2.11}$$

Note the first term in Eq. (2.11) is the same as Eq. (2.6). But the second term is out-of-phase by $\frac{\pi}{2}$. (Do you understand this last statement? If not come see me.)

1. I will now help you show that the plane-of-polarization of the wave in Eq. (2.11) rotates as a function of time (as opposed to remaining fixed along $\hat{x}$, as Eq. (2.6) does).

   (a) Show that Eq. (2.11) is identical to (there is a hint below):

   $$E = E_0 \; Re \left[\frac{1}{\sqrt{2}} \hat{x} e^{\imath(kz-\omega t)} + \frac{\imath}{\sqrt{2}} \hat{y} e^{\imath(kz-\omega t)} \right] \tag{2.12}$$

   Where $Re\left[\cdots\right]$ stands for the real part of the bracketed term. Hint:

      i. Take Eq. (2.12), and substitute $e^{\imath(\theta)} = \cos\left(\theta\right) + \imath \sin\left(\theta\right)$.

      ii. Take only the real part (as is required by the operation $Re\left[\cdots\right]$. You will obtain Eq. (2.11)

   (b) Consider the following for values of time, $t = \frac{kz}{\omega}$, $t = \frac{kz}{\omega} + \frac{\pi}{4\omega}$, $t = \frac{kz}{\omega} + 2\frac{\pi}{4\omega}$, $t = \frac{kz}{\omega} + 3\frac{\pi}{4\omega}$, $t = \frac{kz}{\omega} + 4\frac{\pi}{4\omega}$. At each value of t, draw the direction of the vector in Eq. (2.12). Do you notice anything special? Does this direction change with time? Can you comment on what you see? [1].

   (c) Like for the case of plane-polarized light, we propose $SG_y^+ \leftrightarrow$ Eq. (2.11) which we call right circularly polarized due to the positive sign. You must have noticed in the previous problem that the rotation occurs in a fashion similar to a right hand screw which is why it is called right circularly polarized light. If you flip the sign in Eq. (2.11) you will see that the rotation is similar to that due to a left hand screw and hence the equation with the flipped sign is called left circularly polarized light.

   (d) We then construct the analogy $SG_y^- \leftrightarrow$ Eq. (2.11) with the sign flipped. This analogy gives us the relations

   $$SG_y^+ = \frac{1}{\sqrt{2}} \left[SG_z^+ + \imath SG_z^- \right] \tag{2.13}$$

[1] For reasons you see in this example, Eq. (2.12) is called *circularly polarized light*

and
$$SG_y^- = \frac{1}{\sqrt{2}}\left[SG_z^+ - \imath SG_z^-\right] \tag{2.14}$$
Notice that the $\imath$ makes its appearance here through Eq. (2.12)

(e) *Complex numbers already !!*

(f) Show that the choice in Eqs. (2.13) and (2.14) makes the states $SG_z^\pm$, $SG_x^\pm$ and $SG_y^\pm$ symmetric. Hint: To answer this question, think about the angle between $SG_z^\pm$ and $SG_x^\pm$. Utilize the connection to x and x′ vectors and try to obtain the angle using "dot" products. Now, what are the angles between $SG_z^\pm$ and $SG_y^\pm$?

(g) Importantly, from Eqs. (2.9), (2.10), (2.13) and (2.14) we note that the states corresponding to SG_z^+ and SG_z^- must form a complete set!! Can you explain why?

So this is a two-dimensional space that can have complex coefficients. Hence, in essence SG experiments are explained by invoking a *four-dimensional real linear vector space*!!

2. The quantity $[a \times SG_z^+ + b \times SG_z^-]$ which is now an arbitrary linear combination of the two vectors with (possibly) complex values for "a" and "b" is called a "spinor", since it is an object associated with the spin of a particle.

3. Spinors form the basis of much what exists now-a-days in the quantum information and quantum computing literatures.

