10.2 Introduction to Conics: Parabolas
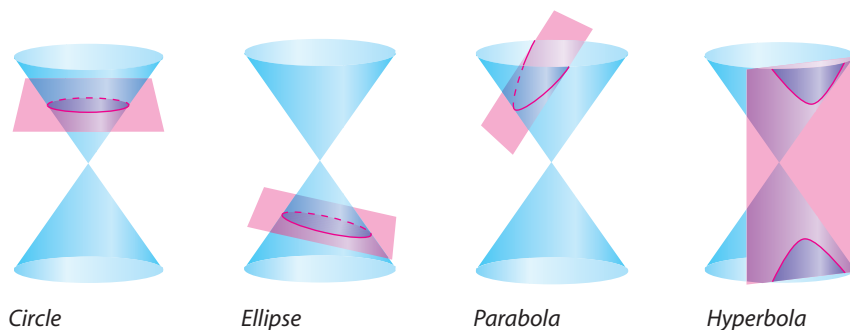
Conics

Conic sections were discovered during the classical Greek period, 600 to 300 B.C. The early Greeks were concerned largely with the geometric properties of conics. It was not until the 17th century that the broad applicability of conics became apparent and played a prominent role in the early development of calculus.

A **conic section** (or simply **conic**) is the intersection of a plane and a double-napped cone. Notice in Figure 10.8 that in the formation of the four basic conics, the intersecting plane does not pass through the vertex of the cone. When the plane does pass through the vertex, the resulting figure is a **degenerate conic,** as shown in Figure 10.9.
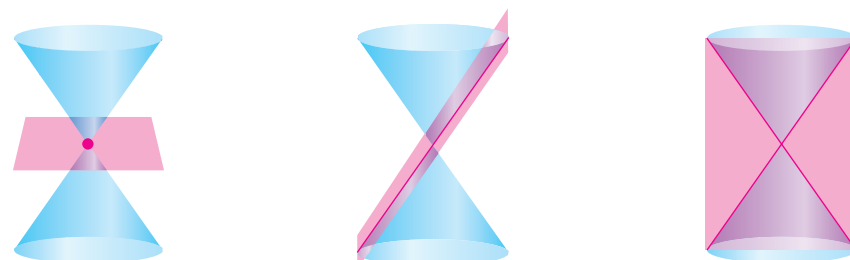

Circle   Ellipse   Parabola   Hyperbola

FIGURE 10.8   *Basic Conics*

Point   Line   Two Intersecting Lines

FIGURE 10.9   *Degenerate Conics*

There are several ways to approach the study of conics. You could begin by defining conics in terms of the intersections of planes and cones, as the Greeks did, or you could define them algebraically, in terms of the general second-degree equation

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0.$$

However, you will study a third approach, in which each of the conics is defined as a **locus** (collection) of points satisfying a geometric property. For example, in Section 1.2, you learned that a circle is defined as the collection of all points (x, y) that are equidistant from a fixed point (h, k). This leads to the standard form of the equation of a circle

$$(x - h)^2 + (y - k)^2 = r^2.$$ Equation of circle

Parabolas

In Section 2.1, you learned that the graph of the quadratic function

$$f(x) = ax^2 + bx + c$$

is a parabola that opens upward or downward. The following definition of a parabola is more general in the sense that it is independent of the orientation of the parabola.

Definition of Parabola

A **parabola** is the set of all points (x, y) in a plane that are equidistant from a fixed line **(directrix)** and a fixed point **(focus)** not on the line.
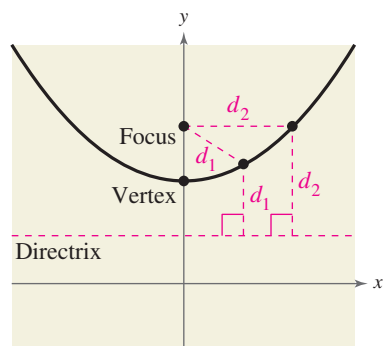
The midpoint between the focus and the directrix is called the **vertex,** and the line passing through the focus and the vertex is called the **axis** of the parabola. Note in Figure 10.10 that a parabola is symmetric with respect to its axis. Using the definition of a parabola, you can derive the following **standard form** of the equation of a parabola whose directrix is parallel to the x-axis or to the y-axis.

FIGURE 10.10   *Parabola*

Standard Equation of a Parabola

The **standard form of the equation of a parabola** with vertex at (h, k) is as follows.

$$(x - h)^2 = 4p(y - k), \ p \neq 0 \qquad \text{Vertical axis, directrix: } y = k - p$$

$$(y - k)^2 = 4p(x - h), \ p \neq 0 \qquad \text{Horizontal axis, directrix: } x = h - p$$

The focus lies on the axis p units (*directed distance*) from the vertex. If the vertex is at the origin $(0, 0)$, the equation takes one of the following forms.

$$x^2 = 4py \qquad \text{Vertical axis}$$

$$y^2 = 4px \qquad \text{Horizontal axis}$$

See Figure 10.11.

For a proof of the standard form of the equation of a parabola, see Proofs in Mathematics on page 807.
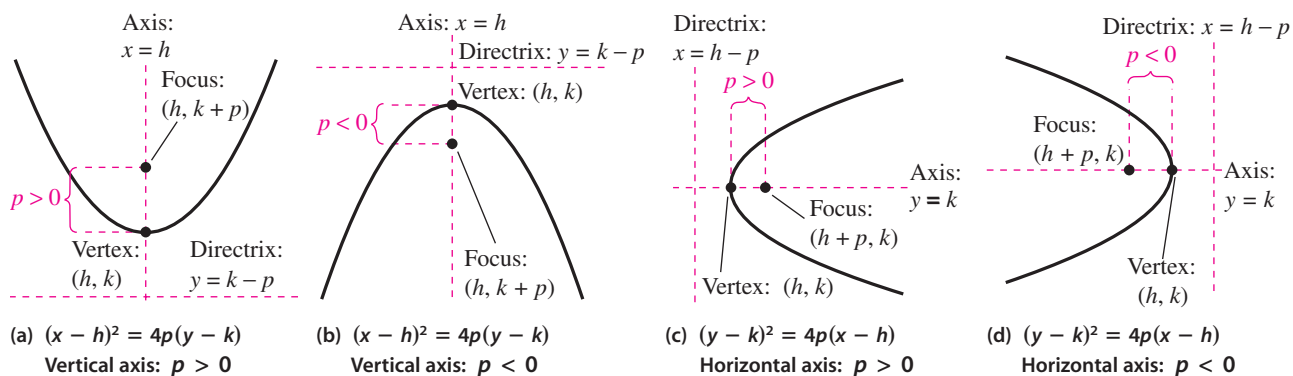

(a) $(x - h)^2 = 4p(y - k)$
   Vertical axis: $p > 0$

(b) $(x - h)^2 = 4p(y - k)$
   Vertical axis: $p < 0$

(c) $(y - k)^2 = 4p(x - h)$
   Horizontal axis: $p > 0$

(d) $(y - k)^2 = 4p(x - h)$
   Horizontal axis: $p < 0$

FIGURE 10.11

Technology

Use a graphing utility to confirm the equation found in Example 1. In order to graph the equation, you may have to use two separate equations:

$$y_1 = \sqrt{8x} \qquad \text{Upper part}$$

and

$$y_2 = -\sqrt{8x}. \qquad \text{Lower part}$$

STUDY TIP

You may want to review the technique of completing the square found in Appendix A.5, which will be used to rewrite each of the conics in standard form.

Example 1   **Vertex at the Origin**

Find the standard equation of the parabola with vertex at the origin and focus $(2, 0)$.

Solution

The axis of the parabola is horizontal, passing through $(0, 0)$ and $(2, 0)$, as shown in Figure 10.12.

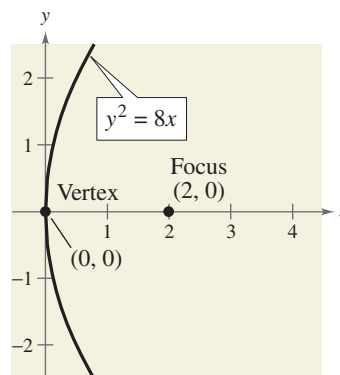FIGURE 10.12

So, the standard form is $y^2 = 4px$, where $h = 0$, $k = 0$, and $p = 2$. So, the equation is $y^2 = 8x$.

✓CHECKPOINT   Now try Exercise 33.

Example 2   **Finding the Focus of a Parabola**

Find the focus of the parabola given by $y = -\frac{1}{2}x^2 - x + \frac{1}{2}$.

Solution

To find the focus, convert to standard form by completing the square.

$$
\begin{aligned}
y &= -\tfrac{1}{2}x^2 - x + \tfrac{1}{2} && \text{Write original equation.}\\
-2y &= x^2 + 2x - 1 && \text{Multiply each side by –2.}\\
1 - 2y &= x^2 + 2x && \text{Add 1 to each side.}\\
1 + 1 - 2y &= x^2 + 2x + 1 && \text{Complete the square.}\\
2 - 2y &= x^2 + 2x + 1 && \text{Combine like terms.}\\
-2(y - 1) &= (x + 1)^2 && \text{Standard form}
\end{aligned}
$$

Comparing this equation with

$$(x - h)^2 = 4p(y - k)$$

you can conclude that $h = -1$, $k = 1$, and $p = -\frac{1}{2}$. Because p is negative, the parabola opens downward, as shown in Figure 10.13. So, the focus of the parabola is $(h, k + p) = \left(-1, \frac{1}{2}\right)$.

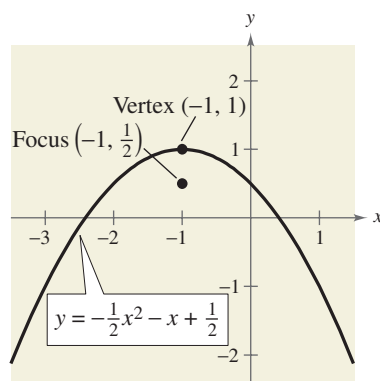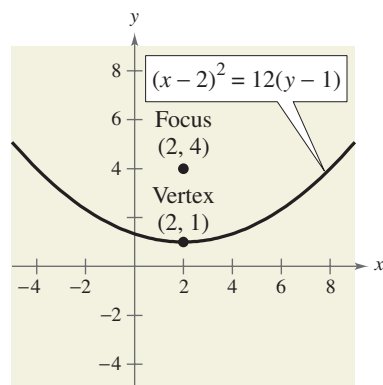✓CHECKPOINT   Now try Exercise 21.

FIGURE 10.13

FIGURE 10.14

Example 3   **Finding the Standard Equation of a Parabola**

Find the standard form of the equation of the parabola with vertex $(2, 1)$ and focus $(2, 4)$.

Solution

Because the axis of the parabola is vertical, passing through $(2, 1)$ and $(2, 4)$, consider the equation

$$(x - h)^2 = 4p(y - k)$$

where $h = 2$, $k = 1$, and $p = 4 - 1 = 3$. So, the standard form is

$$(x - 2)^2 = 12(y - 1).$$

You can obtain the more common quadratic form as follows.

$$(x - 2)^2 = 12(y - 1) \qquad \text{Write original equation.}$$

$$x^2 - 4x + 4 = 12y - 12 \qquad \text{Multiply.}$$

$$x^2 - 4x + 16 = 12y \qquad \text{Add 12 to each side.}$$

$$\frac{1}{12}(x^2 - 4x + 16) = y \qquad \text{Divide each side by 12.}$$

The graph of this parabola is shown in Figure 10.14.

CHECKPOINT   Now try Exercise 45.

Application

A line segment that passes through the focus of a parabola and has endpoints on the parabola is called a **focal chord.** The specific focal chord perpendicular to the axis of the parabola is called the **latus rectum.**

Parabolas occur in a wide variety of applications. For instance, a parabolic reflector can be formed by revolving a parabola around its axis. The resulting surface has the property that all incoming rays parallel to the axis are reflected through the focus of the parabola. This is the principle behind the construction of the parabolic mirrors used in reflecting telescopes. Conversely, the light rays emanating from the focus of a parabolic reflector used in a flashlight are all parallel to one another, as shown in Figure 10.15.

A line is **tangent** to a parabola at a point on the parabola if the line intersects, but does not cross, the parabola at the point. Tangent lines to parabolas have special properties related to the use of parabolas in constructing reflective surfaces.
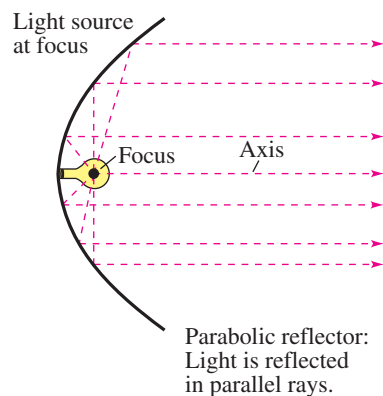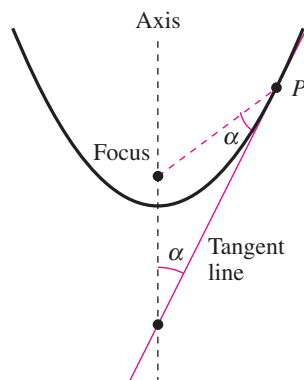

FIGURE 10.15

FIGURE 10.16

Reflective Property of a Parabola

The tangent line to a parabola at a point P makes equal angles with the following two lines (see Figure 10.16).

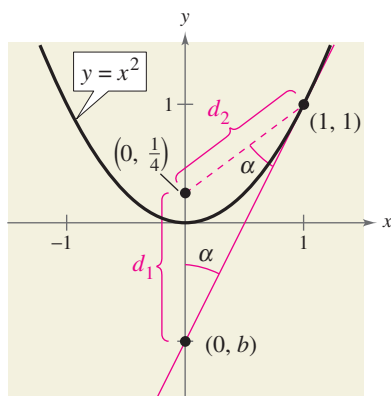1. The line passing through P and the focus

2. The axis of the parabola

Example 4 Finding the Tangent Line at a Point on a Parabola

Find the equation of the tangent line to the parabola given by $y = x^2$ at the point $(1, 1)$.

Solution

For this parabola, $p = \frac{1}{4}$ and the focus is $\left(0, \frac{1}{4}\right)$, as shown in Figure 10.17. You can find the y-intercept $(0, b)$ of the tangent line by equating the lengths of the two sides of the isosceles triangle shown in Figure 10.17:

$$d_1 = \frac{1}{4} - b$$

and

$$d_2 = \sqrt{(1 - 0)^2 + \left[1 - \left(\frac{1}{4}\right)\right]^2} = \frac{5}{4}.$$

Note that $d_1 = \frac{1}{4} - b$ rather than $b - \frac{1}{4}$. The order of subtraction for the distance is important because the distance must be positive. Setting $d_1 = d_2$ produces

$$\frac{1}{4} - b = \frac{5}{4}$$

$$b = -1.$$

So, the slope of the tangent line is

$$m = \frac{1 - (-1)}{1 - 0} = 2$$

and the equation of the tangent line in slope-intercept form is

$$y = 2x - 1.$$

✔**CHECKPOINT**   Now try Exercise 55.

FIGURE **10.17**

Technology

Use a graphing utility to confirm the result of Example 4. By graphing

$$y_1 = x^2 \quad \text{and} \quad y_2 = 2x - 1$$

in the same viewing window, you should be able to see that the line touches the parabola at the point $(1, 1)$.

WRITING ABOUT MATHEMATICS

Television Antenna Dishes Cross sections of television antenna dishes are parabolic in shape. Use the figure shown to write a paragraph explaining why these dishes are parabolic.

Activities

1. Find the vertex, focus, and directrix of the parabola $x^2 - 6x - 4y + 5 = 0$.
    Answer: Vertex $(3, -1)$; Focus $(3, 0)$; Directrix $y = -2$

2. Find the standard form of the equation of the parabola with vertex $(4, 0)$ and directrix $x = 5$.
    Answer: $y^2 = -4(x - 4)$

3. Find an equation of the tangent line to the parabola $y = 2x^2$ at the point $(1, 2)$.
    Answer: $y = 4x - 2$
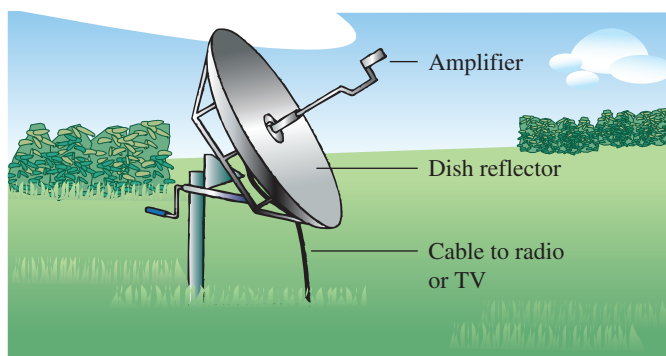
10.2 Exercises

VOCABULARY CHECK: Fill in the blanks.

1. A _____ is the intersection of a plane and a double-napped cone.

2. A collection of points satisfying a geometric property can also be referred to as a _____ of points.

3. A _____ is defined as the set of all points (x, y) in a plane that are equidistant from a fixed line, called the _____, and a fixed point, called the _____, not on the line.

4. The line that passes through the focus and vertex of a parabola is called the _____ of the parabola.

5. The _____ of a parabola is the midpoint between the focus and the directrix.

6. A line segment that passes through the focus of a parabola and has endpoints on the parabola is called a _____ _____ .

7. A line is _____ to a parabola at a point on the parabola if the line intersects, but does not cross, the parabola at the point.

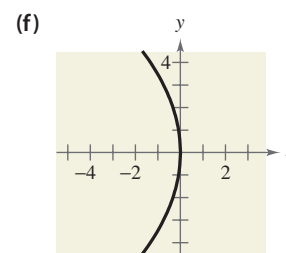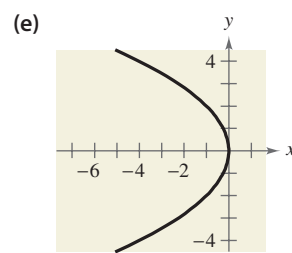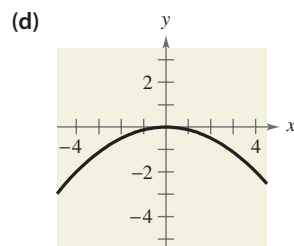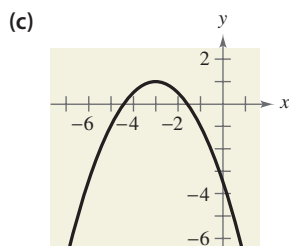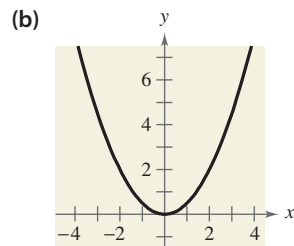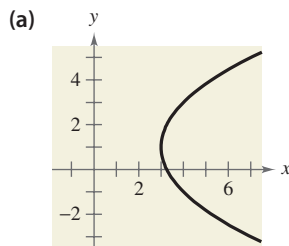**PREREQUISITE SKILLS REVIEW:** Practice and review algebra skills needed for this section at **www.Eduspace.com.**

In Exercises 1–4, describe in words how a plane could intersect with the double-napped cone shown to form the conic section.

1. Circle

2. Ellipse

3. Parabola

4. Hyperbola

In Exercises 5–10, match the equation with its graph. [The graphs are labeled (a), (b), (c), (d), (e), and (f).]

(a)

(b)

(c)

(d)

(e)

(f)

5. $y^2 = -4x$

6. $x^2 = 2y$

7. $x^2 = -8y$

8. $y^2 = -12x$

9. $(y - 1)^2 = 4(x - 3)$

10. $(x + 3)^2 = -2(y - 1)$

In Exercises 11–24, find the vertex, focus, and directrix of the parabola and sketch its graph.

11. $y = \frac{1}{2}x^2$

12. $y = -2x^2$

13. $y^2 = -6x$

14. $y^2 = 3x$

15. $x^2 + 6y = 0$

16. $x + y^2 = 0$

17. $(x - 1)^2 + 8(y + 2) = 0$

18. $(x + 5) + (y - 1)^2 = 0$

19. $\left(x + \frac{3}{2}\right)^2 = 4(y - 2)$

20. $\left(x + \frac{1}{2}\right)^2 = 4(y - 1)$

21. $y = \frac{1}{4}(x^2 - 2x + 5)$

22. $x = \frac{1}{4}(y^2 + 2y + 33)$

23. $y^2 + 6y + 8x + 25 = 0$
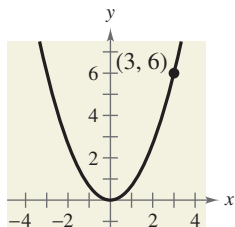
24. $y^2 - 4y - 4x = 0$

In Exercises 25–28, find the vertex, focus, and directrix of the parabola. Use a graphing utility to graph the parabola.
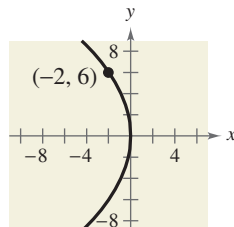
25. $x^2 + 4x + 6y - 2 = 0$

26. $x^2 - 2x + 8y + 9 = 0$

27. $y^2 + x + y = 0$

28. $y^2 - 4x - 4 = 0$

In Exercises 29–40, find the standard form of the equation of the parabola with the given characteristic(s) and vertex at the origin.

29.

30.

31. Focus: $\left(0, -\frac{3}{2}\right)$

32. Focus: $\left(\frac{5}{2}, 0\right)$

33. Focus: $(-2, 0)$

34. Focus: $(0, -2)$

35. Directrix: $y = -1$

36. Directrix: $y = 3$
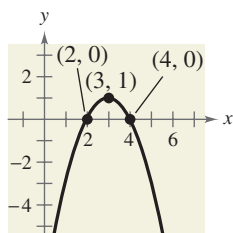
37. Directrix: $x = 2$

38. Directrix: $x = -3$

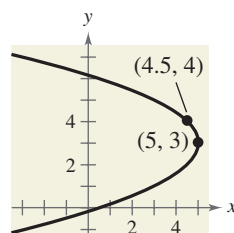**39.** Horizontal axis and passes through the point $(4, 6)$

40. Vertical axis and passes through the point $(-3, -3)$

In Exercises 41–50, find the standard form of the equation of the parabola with the given characteristics.
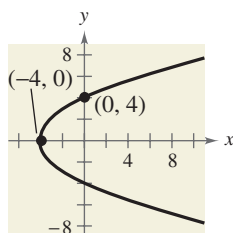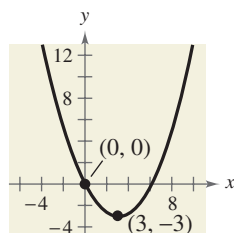
41.

42.

43.

44.

45. Vertex: $(5, 2)$; focus: $(3, 2)$

46. Vertex: $(-1, 2)$; focus: $(-1, 0)$

47. Vertex: $(0, 4)$; directrix: $y = 2$

48. Vertex: $(-2, 1)$; directrix: $x = 1$

49. Focus: $(2, 2)$; directrix: $x = -2$

50. Focus: $(0, 0)$; directrix: $y = 8$

In Exercises 51 and 52, change the equation of the parabola so that its graph matches the description.

51. $(y - 3)^2 = 6(x + 1)$; upper half of parabola

52. $(y + 1)^2 = 2(x - 4)$; lower half of parabola

In Exercises 53 and 54, the equations of a parabola and a tangent line to the parabola are given. Use a graphing utility to graph both equations in the same viewing window. Determine the coordinates of the point of tangency.

Parabola	*Tangent Line*
53. $y^2 - 8x = 0$	$x - y + 2 = 0$
54. $x^2 + 12y = 0$	$x + y - 3 = 0$

In Exercises 55–58, find an equation of the tangent line to the parabola at the given point, and find the *x*-intercept of the line.

55. $x^2 = 2y$, $(4, 8)$

56. $x^2 = 2y$, $\left(-3, \frac{9}{2}\right)$

57. $y = -2x^2$, $(-1, -2)$

58. $y = -2x^2$, $(2, -8)$

59. *Revenue* The revenue R (in dollars) generated by the sale of x units of a patio furniture set is given by

$$(x - 106)^2 = -\frac{4}{5}(R - 14{,}045).$$

Use a graphing utility to graph the function and approximate the number of sales that will maximize revenue.

60. *Revenue* The revenue R (in dollars) generated by the sale of x units of a digital camera is given by

$$(x - 135)^2 = -\frac{5}{7}(R - 25{,}515).$$

Use a graphing utility to graph the function and approximate the number of sales that will maximize revenue.

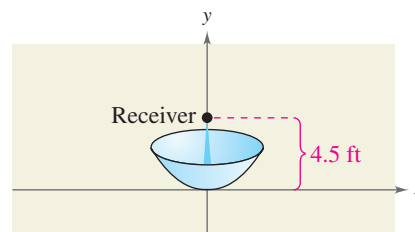**61. *Satellite Antenna*** The receiver in a parabolic television dish antenna is 4.5 feet from the vertex and is located at the focus (see figure). Write an equation for a cross section of the reflector. (Assume that the dish is directed upward and the vertex is at the origin.)

Model It

62. Suspension Bridge Each cable of the Golden Gate Bridge is suspended (in the shape of a parabola) between two towers that are 1280 meters apart. The top of each tower is 152 meters above the roadway. The cables touch the roadway midway between the towers.

(a) Draw a sketch of the bridge. Locate the origin of a rectangular coordinate system at the center of the roadway. Label the coordinates of the known points.

(b) Write an equation that models the cables.

(c) Complete the table by finding the height y of the suspension cables over the roadway at a distance of x meters from the center of the bridge.

Distance, x	Height, y
0	
250	
400	
500	
1000	

63. Road Design Roads are often designed with parabolic surfaces to allow rain to drain off. A particular road that is 32 feet wide is 0.4 foot higher in the center than it is on the sides (see figure).

32 ft   0.4 ft

Not drawn to scale
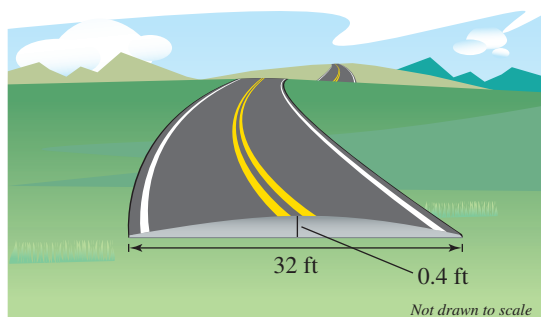
Cross section of road surface

(a) Find an equation of the parabola that models the road surface. (Assume that the origin is at the center of the road.)

(b) How far from the center of the road is the road surface 0.1 foot lower than in the middle?

64. Highway Design Highway engineers design a parabolic curve for an entrance ramp from a straight street to an interstate highway (see figure). Find an equation of the parabola.
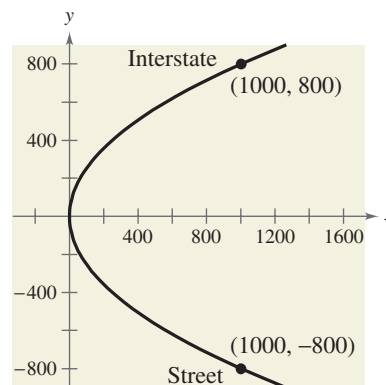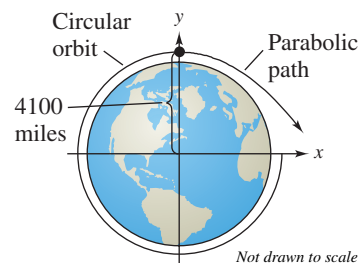

FIGURE FOR **64**

65. Satellite Orbit A satellite in a 100-mile-high circular orbit around Earth has a velocity of approximately 17,500 miles per hour. If this velocity is multiplied by $\sqrt{2}$, the satellite will have the minimum velocity necessary to escape Earth's gravity and it will follow a parabolic path with the center of Earth as the focus (see figure).

(a) Find the escape velocity of the satellite.

(b) Find an equation of the parabolic path of the satellite (assume that the radius of Earth is 4000 miles).

66. Path of a Softball The path of a softball is modeled by

$$-12.5(y - 7.125) = (x - 6.25)^2$$

where the coordinates x and y are measured in feet, with $x = 0$ corresponding to the position from which the ball was thrown.

(a) Use a graphing utility to graph the trajectory of the softball.

(b) Use the *trace* feature of the graphing utility to approximate the highest point and the range of the trajectory.

Projectile Motion **In Exercises 67 and 68, consider the path of a projectile projected horizontally with a velocity of v feet per second at a height of s feet, where the model for the path is**

$$x^2 = -\frac{v^2}{16}(y - s).$$

In this model (in which air resistance is disregarded), y is the height (in feet) of the projectile and x is the horizontal distance (in feet) the projectile travels.

67. A ball is thrown from the top of a 75-foot tower with a velocity of 32 feet per second.

    (a) Find the equation of the parabolic path.

    (b) How far does the ball travel horizontally before striking the ground?

68. A cargo plane is flying at an altitude of 30,000 feet and a speed of 540 miles per hour. A supply crate is dropped from the plane. How many *feet* will the crate travel horizontally before it hits the ground?

Synthesis

True or False? **In Exercises 69 and 70, determine whether the statement is true or false. Justify your answer.**

69. It is possible for a parabola to intersect its directrix.

70. If the vertex and focus of a parabola are on a horizontal line, then the directrix of the parabola is vertical.
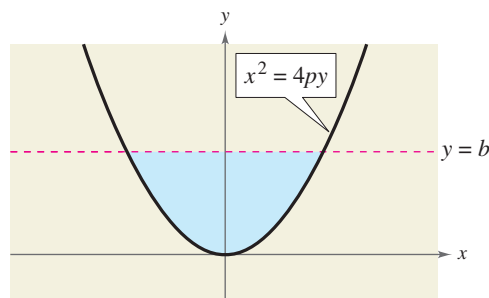
71. *Exploration* Consider the parabola $x^2 = 4py$.

    (a) Use a graphing utility to graph the parabola for $p = 1$, $p = 2$, $p = 3$, and $p = 4$. Describe the effect on the graph when p increases.

    (b) Locate the focus for each parabola in part (a).

    (c) For each parabola in part (a), find the length of the chord passing through the focus and parallel to the directrix (see figure). How can the length of this chord be determined directly from the standard form of the equation of the parabola?
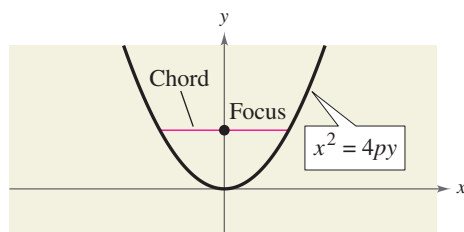
    (d) Explain how the result of part (c) can be used as a sketching aid when graphing parabolas.

72. *Geometry* The area of the shaded region in the figure is

$$A = \frac{8}{3}p^{1/2}b^{3/2}.$$

    (a) Find the area when $p = 2$ and $b = 4$.

    (b) Give a geometric explanation of why the area approaches 0 as p approaches 0.

73. *Exploration* Let (x_1, y_1) be the coordinates of a point on the parabola $x^2 = 4py$. The equation of the line tangent to the parabola at the point is

$$y - y_1 = \frac{x_1}{2p}(x - x_1).$$

What is the slope of the tangent line?

74. *Writing* In your own words, state the reflective property of a parabola.

Skills Review

In Exercises 75–78, list the possible rational zeros of f given by the Rational Zero Test.

75. $f(x) = x^3 - 2x^2 + 2x - 4$

76. $f(x) = 2x^3 + 4x^2 - 3x + 10$

77. $f(x) = 2x^5 + x^2 + 16$

78. $f(x) = 3x^3 - 12x + 22$

79. Find a polynomial with real coefficients that has the zeros 3, $2 + i$, and $2 - i$.

80. Find all the zeros of

$$f(x) = 2x^3 - 3x^2 + 50x - 75$$

if one of the zeros is $x = \frac{3}{2}$.

81. Find all the zeros of the function

$$g(x) = 6x^4 + 7x^3 - 29x^2 - 28x + 20$$

if two of the zeros are $x = \pm 2$.

82. Use a graphing utility to graph the function given by

$$h(x) = 2x^4 + x^3 - 19x^2 - 9x + 9.$$

Use the graph to approximate the zeros of h.

In Exercises 83–90, use the information to solve the triangle. Round your answers to two decimal places.

83. $A = 35°$, $a = 10$, $b = 7$

84. $B = 54°$, $b = 18$, $c = 11$

85. $A = 40°$, $B = 51°$, $c = 3$

86. $B = 26°$, $C = 104°$, $a = 19$

87. $a = 7$, $b = 10$, $c = 16$

88. $a = 58$, $b = 28$, $c = 75$

89. $A = 65°$, $b = 5$, $c = 12$

90. $B = 71°$, $a = 21$, $c = 29$